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Correlations of Observables in Chaotic States of Macroscopic Quantum Systems

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We study correlations of observables in energy eigenstates of chaotic systems of a large size N. We show that the bipartite entanglement of two subsystems is quite strong, whereas macroscopic entanglement of the total system is absent. It is also found that correlations, either quantum or classical, among fewer than $N/2$ points are quite small. These results imply that chaotic states are stable. Invariance of these properties under local operations is also shown.

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It is generally believed that almost all macroscopic systems have chaotic dynamics, because otherwise thermodynamics would not hold. However, properties of chaotic quantum states in macroscopic systems are poorly understood as compared with those in systems of small degrees of freedom.[1] Among such properties, correlations of observables at different points are of particular interest. For example, such correlations are directly related to entanglement,[2] which is the central subject of quantum information theory.[3] The correlations also define the 'cluster property', which is one of the most fundamental notions of field theory.[4] Moreover, the correlations determine the magnitudes of fluctuations of 'additive operators' (see below), which are macroscopic observables. Furthermore, Shimizu and Miyadera[5] (hereafter referred to as SM) showed that two-point correlations determine the stabilities against classical noises, perturbations from environments, and local measurements. In particular, they showed that states with 'macroscopic entanglement' are unstable. Since chaotic dynamics is generally believed to promote entanglement production,[6] and since classical chaos is characterized by extreme sensitivity to the initial condition, it might be tempting to conjecture that chaotic states would be unstable. However, as we will show, such a naive expectation is wrong. In this work, we study these points for chaotic quantum states in macroscopic systems, i.e., systems with a large but finite degrees of freedom.

Two-point correlations and fluctuations of additive operators: We consider an energy eigenstate $|\Psi\rangle$ of a quantum chaotic system which is composed of N ($\gg 1$) sites, where the Hilbert space is the tensor product of local Hilbert spaces. Let $\{\hat{a}_\alpha(l)\}$ be a basis of local observables at site l. Assuming that $\hat{a}_\alpha(l)$'s are bounded, we normalize them as $\mathrm{Tr}[\hat{a}^\dagger_\alpha(l)\hat{a}_\beta(l)] = \text{constant} \times \delta_{\alpha\beta}$. Then, all information about the two-point correlations in $|\Psi\rangle$ are included in the variance–covariance matrix (VCM), $V_{\alpha l,\beta l'} \equiv \langle\Psi|\delta\hat{a}^\dagger_\alpha(l)\,\delta\hat{a}_\beta(l')|\Psi\rangle$, where $\delta\hat{a}_\alpha(l) \equiv \hat{a}_\alpha(l) - \langle\Psi|\hat{a}_\alpha(l)|\Psi\rangle$.[7] This matrix also provides information about fluctuations $\langle\Psi|\delta\hat{A}^\dagger\delta\hat{A}|\Psi\rangle$ of 'additive operators' $\hat{A}$, which are defined as the sums of local operators, $\hat{A} = \sum_{\alpha,l} c_{\alpha l}\hat{a}_\alpha(l)$. Here, $c_{\alpha l}$'s are c-numbers which are asymptotically independent of N. Without loss of generality, we normalize them here as $\sum_{\alpha,l}|c_{\alpha l}|^2 = N$. As shown in ref. 7 the maximum fluctuation of the additive operators is $Ne_{\max}$, where $e_{\max}$ is the maximum eigenvalue of the VCM. For example, $e_{\max} = O(1)$[8] for any product state $|\Psi\rangle = \bigotimes_l |\psi_l\rangle$, whereas $e_{\max} = o(N)$ for 'vacuum states' of many-body physics in accordance with thermodynamics.[5,9] Interestingly, as special states in large but finite systems, there also exist pure states for which $e_{\max} = O(N)$.[5,7] Such pure states are superpositions of macroscopically distinct states; hence, they are called *macroscopically entangled states*.[5,7,10]

Let us evaluate the VCMs of energy eigenstates in chaotic systems using random matrix theory (RMT). Suppose that an energy eigenstate $|\Psi_\lambda\rangle$, labeled by a single parameter λ, is represented as $|\Psi_\lambda\rangle = \sum_i c_{\lambda i}|i\rangle$ in a basis set $\{|i\rangle\}$. The ensemble averages, denoted by an overline, of products of the coefficients can be calculated easily using RMT.[11] For example, in the $d \times d$ Gaussian unitary ensemble (GUE) or Gaussian orthogonal ensemble (GOE),

$$\overline{|c_{\lambda i}|^2} = \frac{1}{d}, \tag{1}$$

$$\overline{|c_{\lambda i}|^4} = \frac{2+q}{d(d+1+q)}, \tag{2}$$

$$\overline{|c_{\lambda i}|^2|c_{\lambda i'}|^2} = \frac{1}{d(d+1+q)} \quad \text{for } i \neq i', \tag{3}$$

where $q = 0$ and 1 for GUE and GOE, respectively. Up to the fourth order, the other combinations which are not written above vanish. Using these formulae, we can evaluate the averages and variances of the matrix elements of the VCMs. Since the right-hand sides of eqs. (1)–(3) are independent of λ, we hereafter drop the label λ. Furthermore, we denote $\overline{\langle\Psi_\lambda|\cdot|\Psi_\lambda\rangle}$ simply by $\overline{\langle\cdot\rangle}$.

We here present results for the case where each site is a qubit, which is equivalent to a spin-1/2 system. In this case, we can use the Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ as a basis of local observables.[12] Therefore, the VCMs of an N-qubit system are $3N \times 3N$ matrices. It is easy to show that $\overline{\langle\sigma_\alpha(l)\rangle} = 0$ and $\overline{\langle\sigma_\alpha(l)\rangle\langle\sigma_\beta(l')\rangle} = 0$ for $l \neq l'$ ($\alpha, \beta = x, y, z$). With the help of these, the average values of the elements of the VCM are calculated as

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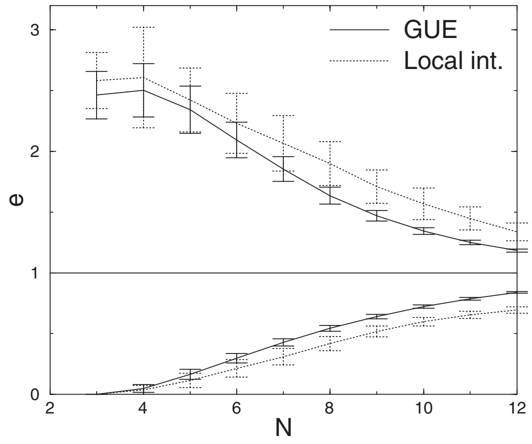

Fig. 1. Maximum and minimum eigenvalues of the VCMs in spin-1/2 systems as functions of system size N. The solid lines represent the results for eigenstates of GUE. The results from GOE, which are not displayed here, are very similar to those of GUE. The dotted lines are the results for the central (2^{N-1}-th) eigenstates of the Hamiltonian given by eq. (8). In both cases, the averages have been taken over 100 samples. The error bars show the standard deviation.

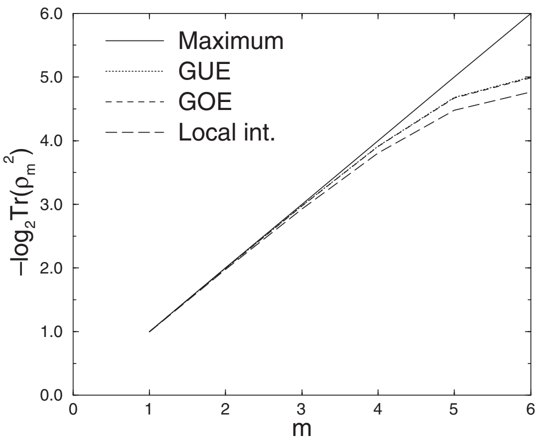

Fig. 2. $-\log_2 \overline{\mathrm{Tr}(\hat{\rho}_m^2)}$ of a subsystem composed of m sites is plotted for $N = 12$. The solid line denotes the upper limit, corresponding to the lower limit of $\overline{\mathrm{Tr}(\hat{\rho}_m^2)}$, for each m. The dotted and dashed lines, which apparently overlap each other in this figure, represent results for GUE and GOE, respectively. The long dashed line shows the result obtained from the central (2^{N-1}-th) eigenvector of the Hamiltonian given by eq. (8).

$$\overline{\langle \delta\sigma_\alpha(l)\delta\sigma_\beta(l')\rangle} = \frac{2^N}{2^N + 1 + q}\delta_{\alpha\beta}\delta_{ll'}. \tag{4}$$

This suggests that the VCM converges to the unit matrix I as $N \to \infty$; hence $e_{\max} = O(1)$. To examine this point, we plot in Fig. 1 the maximum and the minimum eigenvalues, $e_{\max}$ and $e_{\min}$, of the VCMs of energy eigenstates of GUE and GOE as functions of N. It is found that both $e_{\max}$ and $e_{\min}$ (hence all eigenvalues) converge to 1 as N increases, and thus the VCMs approach I. This means that fluctuations of all additive operators are $O(N)$ or less, and that all two-point correlations for $l \neq l'$ are negligibly small. In fact, for $l \neq l'$ we can directly show that

$$\overline{|\langle \sigma_\alpha(l)\sigma_\beta(l')\rangle|^2} = \frac{1+q}{2^N + 1 + q}; \tag{5}$$

hence, $\overline{|\langle \delta\sigma_\alpha(l)\delta\sigma_\beta(l')\rangle|^2}$ is also $O(2^{-N})$. Therefore, *either entanglement or classical correlation will be found to be exponentially small if one observes two arbitrary points of a chaotic state*. Thus, in the sense of SM, chaotic states have the 'cluster property',[13] and are *not* entangled macroscopically. According to the general results of SM, these imply that chaotic states are stable under local measurements, and that the decoherence rate Γ of chaotic states is $O(N)$ or less under weak perturbations from any noise or environment which has short time correlations. This is in sharp contrast to macroscopically entangled states for which some noise gives $\Gamma = O(N^2)$.[5] These points will be discussed later again.

Correlations among many points: We have seen that two-point correlations are infinitesimal. What about m-point correlations for larger m? We can estimate it from bipartite entanglement between two subsystems as follows.

Let us separate the system into two subsystems A and B which contain N_A and $N_B = N - N_A$ sites, respectively. The Hilbert space of subsystem A (B) has dimension $d_A = 2^{N_A}$ ($d_B = 2^{N_B}$). We assume that $N_A \leq N_B$, and evaluate the 'purity' $\mathrm{Tr}(\hat{\rho}_A^2)$ as a measure of bipartite entanglement, where $\hat{\rho}_A$ is the reduced density operator of A; $\hat{\rho}_A \equiv \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$.[14] The purity takes the maximum value 1 when the state $|\Psi\rangle$ is separable (unentangled), and the minimum value $1/d_A$ when $\hat{\rho}_A$ is a scalar matrix I/d_A. For GUE and GOE, eqs. (1)–(3) yield the average purity as[15]

$$\overline{\mathrm{Tr}(\hat{\rho}_A^2)} = \frac{d_A + d_B + q}{d_A d_B + 1 + q} = \frac{1}{d_A}\left(1 + \frac{1}{2^{\Delta N}} + \cdots\right), \tag{6}$$

where $\Delta N \equiv N_B - N_A$ (≥ 0), and '$\cdots$' denotes smaller terms. It is found that the average purity is quite small, which means that *the bipartite entanglement of two subsystems is strong*. With increasing ΔN, in particular, the average purity exponentially approaches the minimum value $1/d_A$. Therefore, $\hat{\rho}_A$ converges to I/d_A exponentially with increasing ΔN, for almost all states. This is demonstrated for $N = 12$ in Fig. 2, in which $\hat{\rho}_m = \hat{\rho}_A$ and $m = N_A$. It is noteworthy that the bipartite entanglement is large but not maximum when $N_A = N_B$.

Suppose now that observables $\hat{a}_1(l_1), \hat{a}_2(l_2), \ldots, \hat{a}_m(l_m)$ at m different points (sites), $l_1, l_2, \ldots, l_m$, are measured. Since these observables commute with each other, their correlations can be calculated in a manner similar to the case of classical stochastic variables. In particular, the cumulants are given by

$$\langle \hat{a}_1(l_1), \ldots, \hat{a}_m(l_m)\rangle_c = (-1)^m \frac{\partial^m \ln Z}{\partial J_1, \ldots, \partial J_m}, \tag{7}$$

where $Z(\{J_i\}) \equiv \langle\Psi| \exp[-\sum_i J_i \hat{a}_i(l_i)]|\Psi\rangle$ is the generator of moments. If we regard the set of m sites $l_1, l_2, \ldots, l_m$ as subsystem A, we have $Z(\{J_i\}) = \mathrm{Tr}(\hat{\rho}_m \exp[-\sum_{i=1}^m J_i \hat{a}_i(l_i)])$, where $\hat{\rho}_m = \hat{\rho}_A$. When $4^{m-N/2} \ll 1$, i.e., when $2^m \ll 2^{N-m}$, $\rho_m = I/2^m$ to a good approximation, according to our results above. In this case, we obtain $\langle \hat{a}_1(l_1), \ldots, \hat{a}_m(l_m)\rangle_c = 0$, which means that there is no correlation among $a_1(l_1), \ldots, a_m(l_m)$. Since they are arbitrary observables at m points, it is concluded that *neither entanglement nor classical correlation can be found if* $4^{m-N/2} \ll 1$, i.e., if the number of points m at which one performs measurements is somewhat smaller than half the total number of points.

On the other hand, if observables at all points are measured, one can detect all types of entanglement. For example, one can detect the bipartite entanglement by measuring a correlation of the Clauser–Horne–Shimony–Holt (CHSH) type.[16] This correlation becomes almost maximum for chaotic states, because we have already shown that the bipartite entanglement between subsystems A and B is almost maximal (when $2^{\Delta N} \gg 1$). However, the enhancement of the CHSH correlation over that of separable states (or of local classical theories) is at most a factor of $\sqrt{2}$. Recently, another correlation of local observables was proposed,[10] by which one can detect macroscopic entanglement in general states (either pure or mixed). It becomes as large as $O(N^2)$ for macroscopically entangled states, whereas it is at most $O(N)$ for states with $e_{\max} = O(1)$ (and for any mixture of such states). For chaotic states, this correlation is at most $O(N)$ because we have shown that $e_{\max} = O(1)$.

Model with short-range interactions: We have assumed RMT in the above analysis. One might suspect that RMT would not be fully applicable to macroscopic systems, in which interactions are short-ranged, because RMT assumes that strengths of interactions are of the same order between any two points. To explore this point, we study eigenstates of the following Hamiltonian:

$$\begin{aligned} H = J\sum_{l=1}^{N} \big\{ &\sigma_x(l)\sigma_x(l+1) + \sigma_z(l)\sigma_z(l+1) \\ &+ \sqrt{2}\cos\phi_l\,\sigma_y(l)\sigma_y(l+1)\big\} \\ &- h\sum_{l=1}^{N}\{\sin\theta_l\,\sigma_x(l) + \cos\theta_l\,\sigma_z(l)\}. \end{aligned} \tag{8}$$

Here, J and h are constants, $\{\phi_l\}$ and $\{\theta_l\}$ are random variables with $0 \le \phi_l, \theta_l < 2\pi$, and $\sigma_\alpha(N+1) = \sigma_\alpha(1)$. This Hamiltonian describes a one-dimensional spin system in which spins interact only with their nearest neighbors, where the interaction in the y-direction is random. Besides, there is an external magnetic field with strength h, whose direction is random in the x–z plane. When J and h are sufficiently large (*e.g.*, $J = h = 1$), this system becomes chaotic, except for states around the lower and upper limits of the energy spectrum, in the sense that the level spacing distribution coincides with that of GOE.

The dotted lines in Fig. 1 represent the maximum and minimum eigenvalues of the VCM for the 2^{N-1}-th eigenstate, which is located at the center of the spectrum, where the chaotic nature appears most clearly. It is seen that the results agree fairly well with those of RMT. Furthermore, the long dashed lines in Fig. 2 represent $-\log_2 \overline{\mathrm{Tr}(\hat{\rho}_m^2)}$ for the same state. The results are close to those for RMT, particularly when m is small. Similar results are obtained also for other eigenstates except those near the upper and lower ends of the spectrum. Since the density of states has a dominant peak at the center of the spectrum, our conclusions from RMT hold for the great majority of eigenstates of the Hamiltonian given by eq. (8).[17] We thus see that RMT correctly describes correlations in energy eigenstates of the chaotic system with short-range interactions.

Invariance under local operations: What happens if local operations are performed on a chaotic state? Let $\{|i_l\rangle\}$ be a basis of the local Hilbert space at site l. An energy eigenstate $|\Psi^{(N)}\rangle$ of an N-site system can be expanded as

$$|\Psi^{(N)}\rangle = \sum_{i_1}\sum_{i_2}\cdots\sum_{i_N} c^{(N)}_{i_1 i_2\cdots i_N}|i_1\rangle|i_2\rangle\cdots|i_N\rangle. \tag{9}$$

For GUE, the probability density of the 2^N coefficients $c^{(N)}_{i_1 i_2\cdots i_N}$ is given by[11]

$$\frac{(2^N-1)!}{\pi^{2^N}}\,\delta\!\left(\sum_{i_1}\cdots\sum_{i_N}|c^{(N)}_{i_1\cdots i_N}|^2 - 1\right). \tag{10}$$

Since this density is invariant under changes of the local basis $\{|i_l\rangle\}$ for any l, the statistical properties of $|\Psi^{(N)}\rangle$ are not changed by local unitary transformations. Furthermore, when a local projective measurement that diagonalizes $\{|i_N\rangle\}$ is performed on the N-th qubit, the post-measurement state (for each measurement) is given by $|i_N\rangle\sum_{i_1}\cdots\sum_{i_{N'}} c^{(N')}_{i_1\cdots i_{N'}} |i_1\rangle\cdots|i_{N'}\rangle \equiv |i_N\rangle|\Psi^{(N')}\rangle$, where $N' \equiv N-1$, and $c^{(N')}_{i_1\cdots i_{N'}} \equiv c^{(N)}_{i_1\cdots i_{N'} i_N}/(\sum_{i_1}\cdots\sum_{i_{N'}}|c^{(N)}_{i_1\cdots i_{N'} i_N}|^2)^{1/2}$. It is easy to show that $\{c^{(N')}_{i_1\cdots i_{N'}}\}$ also obeys the probability distribution of GUE, i.e., eq. (10) with N replaced by N'. Therefore, all the results for $|\Psi^{(N)}\rangle$ hold for $|\Psi^{(N')}\rangle$ as well, with N replaced by N'.[18] The same can be said when a local projective measurement that diagonalizes another local basis set $\{|j_N\rangle\}$ is performed on the N-th qubit, because the density (10) is invariant under changes of the local basis. The same conclusions can be derived for GOE as well. We therefore conclude that *the properties of chaotic states which we have found in this work are invariant under 'local operations,'* including local unitary transformations and local projective measurements diagonalizing a local basis. This implies, for example, that a chaotic state cannot be disentangled by local measurements at less than N points. This is in sharp contrast to, *e.g.*, a 'cat state,' $|\mathrm{C}\rangle \equiv |\uparrow\uparrow\cdots\uparrow\rangle/\sqrt{2} + |\downarrow\downarrow\cdots\downarrow\rangle/\sqrt{2}$, for which a single local measurement suffices to disentangle the state.[5] This is consistent with the conclusion, which we have derived above using the theorem by SM, that chaotic states are stable under local measurements because two-point correlations are infinitesimal.

Discussion: If one defines the 'degree' (or 'strength') $\mathcal{E}$ of entanglement by the minimum number of local operations that are necessary to disentangle it, then, according to our results, $\mathcal{E}$ of chaotic states in large systems is quite high. However, this does *not* mean that they are anomalous as macroscopic states. Indeed, we have shown that they are stable against local measurements and that fluctuations of all additive operators are $O(N)$ or less, in accordance with thermodynamics.[5,7,9] Furthermore, as we have described above, the correlation in ref. 10, which detects macroscopic entanglement, is quite small for chaotic states. Moreover, in the infinite-size limit, all multipoint correlations vanish since the number m of measured points is always finite (hence $m \ll N/2$ as $N \to \infty$). Although the absence of *quantum* correlations for finite m as $N \to \infty$ has been generally proved,[10] we have found here that chaotic states do not even have *classical* correlations in the same limit. Therefore, correlations, either quantum or classical, of chaotic states are neither detectable nor usable in infinite systems.

N dependences: In this work, we have often discussed the dependences on N of various quantities. Unlike a uniform state in a uniform system, however, correspondence between states in systems of different sizes is nontrivial in chaotic systems. For completeness, we finally describe what the N

dependences mean in this paper.

For each system size N, consider a *set* S of all energy eigenstates whose energies are in an interval $(E - \Delta/2, E + \Delta/2]$. To see the N dependence of some quantity Q, look at its (probability) distribution $P_N(Q)$ in S. Then, the N dependence of Q can be discussed in terms of the N dependence of $P_N(Q)$. For example, we say '$Q \leq Q_0$ for almost all states in S for sufficiently large N' if for an arbitrary positive number ϵ there exists N_0 such that $1 - P_N(Q \leq Q_0) \leq \epsilon$ for $N \geq N_0$. We can thus discuss N dependences of various quantities for chaotic states. In particular, we can apply the results of SM, in which N dependences play crucial roles. Note that $P_N(Q)$ is independent of E and Δ in the case of RMT. In the short-range interaction model given by eq. (8), the dependence of $P_N(Q)$ on E and Δ is quite weak if we confine ourselves to states around the center of the energy spectrum. In either case, we can discuss the N dependence without specifying E or Δ.

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8) Throughout this paper, we say that $f(N) = O(g(N))$ if $\lim_{N\to\infty} f(N)/g(N) = \text{const} \neq 0$, and that $f(N) = o(g(N))$ if $\lim_{N\to\infty} f(N)/g(N) = 0$.
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12) Actually we need four matrices $\sigma_x, \sigma_y, \sigma_z$ and I, to represent all 2×2 matrices as their linear combinations. However, we need not consider I because all correlations including it vanish.
13) The cluster property of SM is defined by two-point correlations, whereas that of ref. 4 is defined by multipoint correlations. Clearly, a state having the latter cluster property has the former one as well.
14) For many-body states, there are many types of entanglement and, accordingly, many measures or indices of entanglement. The simplest entanglement is bipartite entanglement, in which a many-body system is regarded as a two-body system composed of two subsystems of the many-body system. For pure states, the von Neumann entropy of the reduced density operator is the unique measure of the bipartite entanglement, whereas the purity is a simplified version of it.
15) The result for GUE, which is a property of random vectors in Hilbert space, has been reported several times in the literature. See, for example, V. M. Kendon, K. Życzkowski and W. J. Munro: Phys. Rev. A **66** (2002) 062310, and references therein.
16) J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt: Phys. Rev. Lett. **23** (1969) 880.
17) Systematic analysis of eigenstates near the ends of the spectrum will be reported elsewhere.
18) $|\Psi^{(N)}\rangle$ is an energy eigenstate of the N-site system, whereas $|\Psi^{(N')}\rangle$ is not necessarily an energy eigenstate of the N'-site system. This difference does not make any difference in the statistical properties.