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Cluster Property and Robustness of Ground States of Interacting Many Bosons

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We study spatial correlation functions of local operators of interacting many bosons confined in a box of a large, but *finite* volume V, for various 'ground states' whose energy densities are almost degenerate. The ground states include the coherent state of interacting bosons (CSIB), the number state of interacting bosons (NSIB), and the number-phase squeezed state of interacting bosons, which interpolates between the CSIB and NSIB. It was shown previously that only the CSIB is robust (i.e., decoheres much more slowly than the other states) against the leakage of bosons into an environment. We show that for the CSIB the spatial correlation of any local operators $A(\boldsymbol{r})$ and $B(\boldsymbol{r}')$ (which are localized around $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively) vanishes as $|\boldsymbol{r}-\boldsymbol{r}'| \sim V^{1/3} \to \infty$, i.e., the CSIB has the 'cluster property.' In contrast, the other ground states do not possess the cluster property. Therefore, we have successfully shown that the robust state has the cluster property. This ensures the consistency of the field theory of bosons with macroscopic theories.

KEYWORDS: cluster property, decoherence, finite system, dissipation, symmetry breaking, thermodynamics, fluctuation, robust, fragile, Bose–Einstein condensation
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It is *required* that microscopic theories should be consistent with macroscopic theories such as thermodynamics. However, it has not been *shown* whether existing microscopic theories of real physical systems are indeed consistent with macroscopic theories. This paper examines one of the consistency conditions,[1)] and show that it is indeed satisfied by a standard microscopic theory of interacting many bosons.

We consider a quantum system of a macroscopic volume V. When the system is closed and $V \to \infty$, the cluster property is one of the most fundamental properties of pure states (pure phases).[2,3)] Here, a state ω is said to have the cluster property[4)] iff

$$|\omega(\hat{A}(\boldsymbol{r})\hat{B}(\boldsymbol{r}')) - \omega(\hat{A}(\boldsymbol{r}))\omega(\hat{B}(\boldsymbol{r}'))| \to 0 \quad \text{as } |\boldsymbol{r}-\boldsymbol{r}'| \to \infty \tag{1}$$

for *any* local operators $\hat{A}(\boldsymbol{r})$ and $\hat{B}(\boldsymbol{r}')$ (at an equal time[5)]) which are localized around $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively. Here, $\omega(\cdot)$ denotes the expectation value in the state ω. Any pure states of infinite systems should have the cluster property.[2,3)] This ensures, for example, that fluctuations of intensive variables are negligible,[2,6–8)] in consistency with thermodynamics. This consistency should be generalized to the case of *finite* V, because thermodynamics is applicable to finite systems as well. In quantum theories of finite systems, one can construct various pure states in a single Hilbert space. However, they are not necessarily consistent with thermodynamics: some states exhibit anomalously large fluctuations of intensive variables.[2,6–8)] Nevertheless, as long as $V < +\infty$, such anomalous states are allowed as pure states in quantum theory, according to the uniqueness of representation for finite systems.[2,3)] (If one simply takes the limit of $V \to \infty$, such states approach states which do not possess the cluster property, hence do not remain pure states.[2,3,6–8)]) Therefore, to be consistent with thermodynamics, there should be some mechanism(s) which excludes anomalous states from pure states in finite systems.

To explore this point, we note that the cluster property and fluctuations are defined only as *static* properties of *closed* systems. However, any finite systems of macroscopic sizes (except for the whole universe) would be interacting with their surrounding environments. Hence, one must also consider *dynamical* properties of *open* systems. In general, the environments induce decoherence, and pure states would evolve into mixed states as time evolves. When some states decohere much faster than the other states, we say such states are *fragile*. When, on the other hand, some states decohere much slower than the other states, we say such states are *robust*. Examples of fragile and robust states were given, *e.g.*, in refs. 9–11, where it was suggested that fragile states would be much difficult to prepare and observe than robust states.[12)]

These observations lead to the following conjecture: it is necessary for a microscopic theory to be consistent with macroscopic theories that *states with anomalous fluctuations are fragile* (hence difficult to prepare and observe), whereas *robust states are non-fluctuating states* (which possess the cluster property in the limit of $V \to \infty$). Although this very fundamental requirement has been suggested in many places (*e.g.*, in refs. 2 and 8), it has not been known yet (to the authors' knowledge) whether general quantum field theories indeed satisfy this requirement. [Conversely, if this is not the case, the fundamental requirement may place a limit on possible forms of field theories in such a way that the requirement is satisfied.] Even for a particular model or theory of a macroscopic system, it was not examined yet whether the requirement is satisfied. From a microscopic point of view, it is non-trivial — may be surprising — that a *dynamical* property (robustness) of an *open* system subject to dissipations is directly related to a *static* property (cluster property) of a *closed* system.

In this paper, we show that the fundamental requirement is indeed satisfied by a standard microscopic theory in condensed matter, i.e., the field theory of bosons with repulsive interaction. For interacting many bosons confined

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in a large but *finite* box, many wave functions had been proposed as the 'ground states,' whose energy densities are almost degenerate (completely degenerate when $V \to \infty$).[11,13] Among them are, *e.g.*, Bogoliubov's ground state, and the ground state that has *exactly* N bosons, which we call the number state of interacting bosons (NSIB).[14] In a previous paper,[11] we already studied the robustness, against leakage of bosons into an environment, of these wave functions. It was shown there that most of them are fragile, and that a robust state is an exceptional wave function which we call the coherent state of interacting bosons (CSIB). Hence, we shall show that the fundamental requirement is satisfied by showing that the CSIB possesses the cluster property (in the limit of $V \to \infty$), whereas the other ground states do not.

For bosons confined in a finite box of volume V with the periodic boundary conditions, the CSIB with amplitude α ($= |\alpha| e^{i\varphi}$) is defined by[11,14]

$$|\alpha, G\rangle \equiv e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} |N, G\rangle, \tag{2}$$

where $|N, G\rangle$ denotes the NSIB. Note that N is well-defined because V is finite, and thus the vector $|N, G\rangle$ exists. The explicit form of $|N, G\rangle$ was given in refs. 14 and 15 in the case of weakly-interacting bosons. Since $|N, G\rangle$ has a complicated wave function, the CSIB is totally different from the coherent state of free bosons, which is defined by $|\alpha, 0\rangle \equiv e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} (\alpha^N/\sqrt{N!}) |N, 0\rangle$, where $|N, 0\rangle$ denotes the number state of *free* bosons, in which N bosons occupy the lowest single-body state whereas the other single-body states are vacant. On the other hand, some properties of the CSIB are similar to those of the coherent state of free bosons; *e.g.*, $\langle N\rangle = \langle \delta N^2\rangle = |\alpha|^2$, hence $\langle \delta N^2\rangle/\langle N\rangle = 1$. Since we are interested in macroscopic systems, for which $\langle N\rangle$ is macroscopically large, we assume $|\alpha| \gg 1$ in the following analysis.

Before going further, a few words are worth mentioning here: (i) The above wave function of the CSIB might look against a superselection rule, which 'forbids' coherent superpositions of states with different numbers of (massive) bosons. However, we previously showed that such superpositions are allowed for a subsystem of a huge system if the wave function of the total system is appropriately taken.[11,16] (ii) The robust state $|\alpha, G\rangle$ is not an energy eigenstate (whereas an energy eigenstate $|N, G\rangle$ is fragile). Hence, even when interactions with the environments are negligible, the CSIB (prepared at $t = 0$) evolves with time (in the Schrödinger picture). Putting $\alpha(t) \equiv e^{i(\varphi - \mu t/\hbar)}|\alpha|$, $\mu \equiv (\partial E_{N,G}/\partial N)_{N=|\alpha|^2}$ (with $E_{N,G}$ being the eigenenergy of $|N, G\rangle$), and $\mathcal{G} \equiv (E_{N,G} - \mu N)_{N=|\alpha|^2}$, we find the time evolution as

$$|\alpha, G; t\rangle = e^{-i\mathcal{G}t/\hbar} \, |\alpha(t), G\rangle, \tag{3}$$

where terms of order $1/V$ have been neglected. These neglected terms cause a *spontaneous* collapse of the wave function,[13,17] i.e., a collapse through the *internal* dynamics (without perturbations from an environment). The collapse time t_{coll} increases with V (*e.g.*, $t_{coll} \propto V^{1/2}$ for a uniform system[13,17]), and becomes of a macroscopic time scale for a large V. In this work, *we are interested in a much shorter time range* $0 \le t \ll t_{coll}$. Hence, the $\mathcal{O}(1/V)$ terms can be neglected in the derivation of eq. (3), and the CSIB does not collapse spontaneously; the time evolution only induces the phase rotations of α. (The overall phase factor $e^{-i\mathcal{G}t/\hbar}$ has no physical meaning.) Therefore, it is sufficient to examine the cluster property at $t = 0$ for a general value of α ($|\alpha| \gg 1$). This should be contrasted with the coherent state of free bosons: it collapses spontaneously due to boson-boson interactions, and $t_{coll} = \mathcal{O}(V^0)$. As a result, although it has the cluster property at $t = 0$, the property will be lost when $t = \mathcal{O}(V^0)$. In this paper, we are not interested in such states, and we will only consider the states (such as the CSIB and NSIB) which are stable enough [i.e., t_{coll} is much longer than $\mathcal{O}(V^0)$] against the internal dynamics.

To examine the cluster property, we decompose the boson field $\hat{\psi}$ into the anomalous and regular parts:[11,14,18]

$$\hat{\psi} = \hat{\Xi} + \hat{\psi}'. \tag{4}$$

Here, $\hat{\Xi}$ is the anomalous part, which connects the ground states of different numbers of bosons as

$$\hat{\Xi}|N, G\rangle = \sqrt{N}\xi |N-1, G\rangle, \tag{5}$$

where $\xi \equiv \langle N-1, G|\hat{\psi}|N, G\rangle/\sqrt{N}$. The magnitudes of ξ characterizes the condensation,[19] and we here consider the condensed states for which

$$\sqrt{N}\xi = \mathcal{O}(1). \tag{6}$$

Note that the CSIB is an eigenstate of $\hat{\Xi}$, i.e., $\hat{\Xi}|\alpha, G\rangle = \alpha\xi|\alpha, G\rangle$.[11,14] On the other hand, the regular part $\hat{\psi}'$ transforms $|N, G\rangle$ into (a superposition of) excited states (of $N-1$ bosons):

$$\langle N - \Delta N, G|\hat{\psi}'|N, G\rangle = 0, \quad \text{hence } \langle \beta, G|\hat{\psi}'|\alpha, G\rangle = 0. \tag{7}$$

Note however that $\hat{\psi}'|N, G\rangle \neq 0$ and $\hat{\psi}'|\alpha, G\rangle \neq 0$. For example,

$$\int \langle \alpha, G|\hat{\psi}'^{\dagger}\hat{\psi}'|\alpha, G\rangle d^3r = \langle N\rangle - \langle N_0\rangle \neq 0, \tag{8}$$

where $\langle N_0\rangle$ is the so-called "number of the condensate particles".[14,18] For weakly-interacting bosons, we can show by explicit calculations that

$$\langle N, \nu|[\hat{\Xi}, \hat{\Xi}^{\dagger}]|N', \nu'\rangle, \ \langle N, \nu|[\hat{\Xi}, \hat{\psi}']|N', \nu'\rangle, \langle N, \nu|[\hat{\Xi}, \hat{\psi}'^{\dagger}]|N', \nu'\rangle \ = s(1/V), \tag{9}$$

where $|N, \nu\rangle$ and $|N', \nu'\rangle$ are energy eigenstates that have exactly N bosons (here, ν and ν' label them; *e.g.*, $|N, \nu\rangle = |N, G\rangle$ for $\nu = G$), and $s(x)$ denotes a smooth function that vanishes as $x \to 0$. Lifshitz and Pitaevskii[18] claimed eq. (9) even for bosons with stronger interactions. If this is the case, the following results are applicable to a wide range of interaction strength.

We first examine the case where the local operators in eq. (1) take the following forms:

$$\hat{A}(\boldsymbol{r}) = \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r})\cdots, \tag{10}$$

$$\hat{B}(\boldsymbol{r}) = \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\cdots. \tag{11}$$

Their correlation for the CSIB is evaluated as

$$\begin{aligned}
&\langle\alpha, \mathrm{G}|\hat{A}(\boldsymbol{r})\hat{B}(\boldsymbol{r}')|\alpha, \mathrm{G}\rangle \\
&= \langle\alpha, \mathrm{G}|\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r})\}\{\hat{\Xi} + \hat{\psi}'(\boldsymbol{r})\}\cdots\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\cdots|\alpha, \mathrm{G}\rangle \\
&= \langle\alpha, \mathrm{G}|(\text{all } \hat{\Xi}^\dagger\text{'s are moved to the left, and all } \hat{\Xi}\text{'s are moved to the right})|\alpha, \mathrm{G}\rangle + s(1/V) \\
&= \langle\alpha, \mathrm{G}|(\text{all } \hat{\Xi}^\dagger\text{'s are replaced with } \alpha^*\xi^*\text{, and all } \hat{\Xi}\text{'s are replaced with } \alpha\xi)|\alpha, \mathrm{G}\rangle + s(1/V) \\
&= \langle\alpha, \mathrm{G}|\{\alpha^*\xi^* + \hat{\psi}'^\dagger(\boldsymbol{r})\}\{\alpha\xi + \hat{\psi}'(\boldsymbol{r})\}\cdots\{\alpha^*\xi^* + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\{\alpha^*\xi^* + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\cdots|\alpha, \mathrm{G}\rangle + s(1/V),
\end{aligned} \tag{12}$$

where use has been made of eq. (9). As mentioned above [eq. (7)], $\hat{\psi}'$ and $\hat{\psi}'^\dagger$ transform the ground states into excited states. Since any excitation cannot propagate a long distance in zero time interval, equal-time correlations of $\hat{\psi}'$ vanish for a large $|\boldsymbol{r} - \boldsymbol{r}'|$. (This fact has been used, *e.g.*, in the standard argument on the ODLRO.[18]) Hence, the last line of eq. (12) approaches, as $|\boldsymbol{r} - \boldsymbol{r}'| \sim V^{1/3}$,

$$\begin{aligned}
&\langle\alpha, \mathrm{G}|\{\alpha^*\xi^* + \hat{\psi}'^\dagger(\boldsymbol{r})\}\{\alpha\xi + \hat{\psi}'(\boldsymbol{r})\}\cdots|\alpha, \mathrm{G}\rangle\langle\alpha, \mathrm{G}|\{\alpha^*\xi^* + \hat{\psi}\prime\dagger(\boldsymbol{r}')\}\{\alpha^*\xi^* + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\cdots|\alpha, \mathrm{G}\rangle + s(1/V) \\
&= \langle\alpha, \mathrm{G}|\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r})\}\{\hat{\Xi} + \hat{\psi}'(\boldsymbol{r})\}\cdots|\alpha, \mathrm{G}\rangle\langle\alpha, \mathrm{G}|\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r}')\}\cdots|\alpha, \mathrm{G}\rangle + s(1/V).
\end{aligned} \tag{13}$$

Therefore,

$$\langle\alpha, \mathrm{G}|\hat{A}(\boldsymbol{r})\hat{B}(\boldsymbol{r}')|\alpha, \mathrm{G}\rangle = \langle\alpha, \mathrm{G}|\hat{A}(\boldsymbol{r})|\alpha, \mathrm{G}\rangle\langle\alpha, \mathrm{G}|\hat{B}(\boldsymbol{r}')|\alpha, \mathrm{G}\rangle + s(1/V) \quad \text{for } |\boldsymbol{r} - \boldsymbol{r}'| \sim V^{1/3}. \tag{14}$$

By similar calculations, we also obtain the relation (14) for other forms of $\hat{A}$ and $\hat{B}$, including those which consist of derivatives of the field operators. Therefore, eq. (1) is indeed satisfied in the limit of $V \to \infty$ (while keeping the boson density $\langle N\rangle/V$ constant) by the CSIB, for any local operators.

More rigorously, we must perform smoothing of the field operators[2,3] because, for example, $\hat{\psi}^\dagger(\boldsymbol{r})|0\rangle$ cannot be a vector in a Hilbert space since its norm diverges like $\lim_{\boldsymbol{r}\to\boldsymbol{0}} \delta(\boldsymbol{r})$, whereas any non-zero vector in a Hilbert space should be normalizable. Let $\mathcal{S}(\boldsymbol{R}^3)$ be a set of smooth functions ($\boldsymbol{R}^3 \mapsto \boldsymbol{C}$) with fast decrease. Using a function $f \in \mathcal{S}(\boldsymbol{R}^3)$, we define

$$\hat{\psi}^\dagger(f) \equiv \int f(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r})\mathrm{d}^3r = C_f\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(f), \tag{15}$$

where

$$C_f \equiv \int f(\boldsymbol{r})\mathrm{d}^3r, \tag{16}$$

$$\hat{\psi}'^\dagger(f) \equiv \int f(\boldsymbol{r})\hat{\psi}'^\dagger(\boldsymbol{r})\mathrm{d}^3r. \tag{17}$$

Using another function $g \in \mathcal{S}(\boldsymbol{R}^3)$, we also define $\hat{\psi}^\dagger(g)$, C_g, and $\hat{\psi}'^\dagger(g)$ in a similar manner. From these definitions and eq. (9), one can easily show that

$$\langle N, \nu|[\hat{\Xi}, \hat{\Xi}^\dagger]|N', \nu'\rangle,\ \langle N, \nu|[\hat{\Xi}, \hat{\psi}'(f)]|N', \nu'\rangle, \langle N, \nu|[\hat{\Xi}, \hat{\psi}'^\dagger(f)]|N', \nu'\rangle\ = s(1/V). \tag{18}$$

Let us construct $\hat{A}(f)$ and $\hat{B}(g)$ from $\hat{\psi}(f)$ and $\hat{\psi}(g)$, respectively, in a manner similar to eqs. (10) and (11). When f and g are centered around $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively, we can show, using eq. (18), by calculations similar to eqs. (12) and (13), that

$$\langle\alpha, \mathrm{G}|\hat{A}(f)\hat{B}(g)|\alpha, \mathrm{G}\rangle = \langle\alpha, \mathrm{G}|\hat{A}(f)|\alpha, \mathrm{G}\rangle\langle\alpha, \mathrm{G}|\hat{B}(g)|\alpha, \mathrm{G}\rangle + s(1/V) \quad \text{for } |\boldsymbol{r} - \boldsymbol{r}'| \sim V^{1/3}. \tag{19}$$

The same relation can also be shown when operators $\hat{A}(f)$ and $\hat{B}(g)$ are constructed from smoothed operators of derivatives of field operators. It is therefore concluded that the CSIB possesses the cluster property in the limit of $V \to \infty$ (while keeping the boson density constant).

On the other hand, the NSIB does not possess the cluster property. We can easily see this by taking $\hat{A}(\boldsymbol{r}) = \hat{\psi}^\dagger(\boldsymbol{r})$ and $\hat{B}(\boldsymbol{r}') = \hat{\psi}(\boldsymbol{r}')$. In fact, as $|\boldsymbol{r} - \boldsymbol{r}'| \sim V^{1/3}$,

$$\begin{aligned}
&\langle N, \mathrm{G}|\hat{A}(\boldsymbol{r})\hat{B}(\boldsymbol{r}')|N, \mathrm{G}\rangle - \langle N, \mathrm{G}|\hat{A}(\boldsymbol{r})|N, \mathrm{G}\rangle\langle N, \mathrm{G}|\hat{B}(\boldsymbol{r}')|N, \mathrm{G}\rangle \\
&= \langle N, \mathrm{G}|\{\hat{\Xi}^\dagger + \hat{\psi}'^\dagger(\boldsymbol{r})\}\{\hat{\Xi} + \hat{\psi}'(\boldsymbol{r}')\}|N, \mathrm{G}\rangle - \langle N, \mathrm{G}|\hat{\psi}^\dagger(\boldsymbol{r})|N, \mathrm{G}\rangle\langle N, \mathrm{G}|\hat{\psi}(\boldsymbol{r}')|N, \mathrm{G}\rangle \\
&= \langle N, \mathrm{G}|\hat{\Xi}^\dagger\hat{\Xi}|N, \mathrm{G}\rangle + s(1/V) \\
&= |\xi|^2N + s(1/V),
\end{aligned} \tag{20}$$

which does not vanish as $V \to \infty$ (while keeping the boson density constant) because $|\xi|^2N = \mathcal{O}(1)$.

As another example, we examine the number-phase squeezed state of interacting bosons (NPIB).[14] Its energy density is almost degenerate (completely degenerate when $V \to \infty$) with the CSIB and NSIB, and its wave function interpolates between these states. The number (phase) uncertainty of the NPIB is larger (smaller) than that of the CSIB, hence the name 'squeezed state'. However, unlike the standard squeezed states, the NPIB has the minimum allowable value of the number-phase uncertainty product, like the CSIB.[14)] The NPIB is characterized by two independent parameters N and ζ, and is defined by

$$|N, \zeta, \mathrm{G}\rangle \equiv \sqrt{K(N, |\zeta|^2)}\mathrm{e}^{-|\zeta|^2/2} \sum_{M=0}^{N} \frac{\zeta^{*(N-M)}}{\sqrt{(N-M)!M!}} |M, \mathrm{G}\rangle, \tag{21}$$

where K is a normalization constant.[14)] We henceforth assume that $N \gg |\zeta|^2 \gg 1$, for which $K = 1$, $\langle N \rangle = N - |\zeta|^2$, and $\langle \delta N^2 \rangle = |\zeta|^2$ to a good approximation.[14)] After lengthy calculations, we can show that, as $|\boldsymbol{r} - \boldsymbol{r}'| \sim V^{1/3}$,

$$\begin{aligned} &\langle N, \zeta, \mathrm{G}|\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}')|N, \zeta, \mathrm{G}\rangle - \langle N, \zeta, \mathrm{G}|\hat{\psi}^\dagger(\boldsymbol{r})|N, \zeta, \mathrm{G}\rangle\langle N, \zeta, \mathrm{G}|\hat{\psi}(\boldsymbol{r}')|N, \zeta, \mathrm{G}\rangle \\ &= |\xi|^2\langle N \rangle/2\langle \delta N^2 \rangle + s(1/V) \\ &\simeq |\xi|^2 N/2|\zeta|^2 + s(1/V). \end{aligned} \tag{22}$$

Since $|\xi|^2 N = \mathcal{O}(1)$ and $|\zeta|^2 \gg 1$ (by assumption), this correlation is small. However, it does not vanish as $V \to \infty$, because ζ is independent of N and V. Therefore, the NPIB does not possess the cluster property, either. Comparing eq. (22) with eq. (20), wee see that the spatial correlation decreases as $|\zeta|^2$ ($= \langle \delta N^2 \rangle$) is increased, i.e, as the NPIB moves from near the NSIB toward the CSIB. Note, however, that eq. (22) is applicable neither the NSIB nor CSIB (because $N \gg |\zeta|^2 \gg 1$ is assumed): our main results are the *set* of eqs. (14), (20) and (22).

We have thus confirmed our conjecture that a robust state has the cluster property. We must be careful in discussing the converse statement — a state with the cluster property is robust — for two reasons. First, when a state has the cluster property at a particular time, it may evolve spontaneously into another state which does not have the cluster property. A typical example is the coherent state of free bosons as mentioned above. Second, excited states generally have finite lifetimes, which tend to be shorter for higher-energy states. This means that higher-energy states decohere quickly. It would be non-trivial to define the robustness for such states. In this work, we have confined ourselves to the ground states.

In conclusion, we have examined the cluster property of various ground states of interacting many bosons confined in a box of a finite volume V. It is shown that the robust ground state (CSIB) possesses the cluster property (in the limit of $V \to \infty$), whereas the fragile ground states (NSIB and NPIB) do not. Namely, the cluster property, which is defined as a *static* property of a *closed* system, is directly related to the robustness, which is a *dynamical* property of an *open* system. This fact, which was *assumed* previously as one of fundamental requirements which ensure consistency of a microscopic theory with macroscopic theories, has been shown to be satisfied by a realistic model of interacting particles for the first time.

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18) E. M. Lifshitz and L. P. Pitaevskii: *Statistical Physics Part II* (Pergamon, New York, 1980) Sect. 26.
19) Strictly speaking, ξ depends weakly on N. Since we can show that this N dependence does not alter our results, eqs. (14) and (19), we here describe the calculations in which this N dependence is neglected.