

Macroscopic entanglement of many-magnon states

Tomoyuki Morimae,^{1,2,*} Ayumu Sugita,^{3,†} and Akira Shimizu^{1,2,‡}

¹*Department of Basic Science, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8902, Japan*

²*PRESTO, JST, 4-1-8 Honcho Kawaguchi, Saitama, Japan*

³*Department of Applied Physics, Osaka City University, 3-3-138 Sugimoto, Osaka 558-8585, Japan*

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We study macroscopic entanglement of various pure states of a one-dimensional N -spin system with $N \gg 1$. Here, a quantum state is said to be macroscopically entangled if it is a superposition of macroscopically distinct states. To judge whether such superposition is hidden in a general state, we use an essentially unique index p : A pure state is macroscopically entangled if $p=2$, whereas it may be entangled but not macroscopically if $p < 2$. This index is directly related to fundamental stabilities of many-body states. We calculate the index p for various states in which magnons are excited with various densities and wave numbers. We find macroscopically entangled states ($p=2$) as well as states with $p=1$. The former states are unstable in the sense that they are unstable against some local measurements. On the other hand, the latter states are stable in the senses that they are stable against any local measurements and that their decoherence rates never exceed $O(N)$ in any weak classical noises. For comparison, we also calculate the von Neumann entropy $S_{N/2}(N)$ of a subsystem composed of $N/2$ spins as a measure of bipartite entanglement. We find that $S_{N/2}(N)$ of some states with $p=1$ is of the same order of magnitude as the maximum value $N/2$. On the other hand, $S_{N/2}(N)$ of the macroscopically entangled states with $p=2$ is as small as $O(\log N) \ll N/2$. Therefore larger $S_{N/2}(N)$ does *not* mean more instability. We also point out that these results are partly analogous to those for interacting many bosons. Furthermore, the origin of the huge entanglement, as measured either by p or $S_{N/2}(N)$, is discussed to be due to spatial propagation of magnons.

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I. INTRODUCTION AND SUMMARY

Many-partite entanglement, i.e., entanglement in a system that is composed of many sites or parties, has been attracting much attention recently [1–22]. It is known that the number of possible measures of entanglement grows dramatically as the number of sites is increased [22]. Different measures are related to different physical properties. Therefore one must specify physical properties of interest in order to determine a proper measure or index.

In this paper, we study *macroscopic entanglement* of various states in a quantum many-spin system. Here, a quantum state is said to be macroscopically entangled if it is a superposition of macroscopically distinct states (see Sec. III A). Although such superposition is trivially recognized for some states (such as the “cat” state), it is hard to find such superposition by intuition for general states. In order to judge whether such superposition is hidden in a general state, we use an essentially unique index p , defined by Eq. (16). A pure state is macroscopically entangled if $p=2$, whereas it may be entangled but not macroscopically if $p < 2$. Unlike many other measures or indices of entanglement, there is an efficient method of computing p for any given states [3].

It was shown by Shimizu and Miyadera [1] (hereafter referred as SM) that this index is directly related to fundamental stabilities of many-body states, i.e., to fragility in

noises or environments and to stability against local measurements. That is, a state with $p=1$ is not particularly unstable against noises in the sense that its decoherence rate does not exceed $O(N)$ in any noises or environments, whereas the decoherence rate of a state with $p=2$ can become as large as $O(N^2)$ [23]. Furthermore, a quantum state with $p=2$ is unstable against local measurements, whereas a homogeneous state with $p=1$ is stable.

We consider a one-dimensional N -spin system with $N \gg 1$, and calculate the index p for various pure states in which magnons are excited with various densities and wave numbers. We find macroscopically entangled states ($p=2$) as well as “normal” states with $p=1$ which *are* entangled but *not* macroscopically. According to SM, they are unstable and stable many-body states, respectively.

For comparison, we also calculate the von Neumann entropy $S_{N/2}(N)$ of a subsystem composed of $N/2$ spins as a measure of bipartite entanglement. We find that some states with $S_{N/2}(N) = O(N)$, which is of the same order of magnitude as the *maximum* value $N/2$, are normal states in the sense that $p=1$. On the other hand, some of other states, which are macroscopically entangled ($p=2$), have much smaller value of $S_{N/2}(N)$ of $O(\log N)$.

These results demonstrate that the degrees of entanglement are totally different if different measures or indices are used. Furthermore, stabilities of quantum states are not simply related to the degrees of entanglement: Different stabilities are related to different measures or indices. In particular, fragility in noises and the stability against local measurements are directly related to p , hence are basically independent of $S_{N/2}(N)$.

*Electronic address: morimae@ASone.c.u-tokyo.ac.jp

†Electronic address: sugita@a-phys.eng.osaka-cu.ac.jp

‡Electronic address: shmz@ASone.c.u-tokyo.ac.jp

The results also demonstrate that states with huge entanglement, as measured by either p or $S_{N/2}(N)$, can be easily constructed by simply applying creation operators of magnons to a ferromagnetic state, which is a separable state. Neither randomness nor elaborate tuning are necessary to construct states with huge entanglement from a separable state. This should be common to most quantum systems: By exciting a macroscopic number of elementary excitations, one can easily construct states with huge entanglement. To generate such states experimentally, however, one must also consider the fundamental stabilities mentioned above: States with $p=2$ would be quite hard to generate experimentally, whereas states with large $S_{N/2}(N)$ would be able to be generated rather easily.

The present paper is organized as follows: In Sec. II, we shortly review physics of magnons, and present state vectors of many-magnon states under consideration. We explain the index p for the macroscopic entanglement, and present an efficient method of computing p in Sec. III. In Sec. IV, we study macroscopic entanglement of many-magnon states by evaluating p . We study their bipartite entanglement in Sec. V for comparison purposes. Stabilities of the many-magnon states are discussed in Sec. VI. In Sec. VII A, we point out that our results are analogous to those for interacting many bosons. Furthermore, we discuss the origin of the huge entanglement in Sec. VII B.

II. MANY-MAGNON STATES

In this section, we briefly review the physics of magnons in order to establish notations. A magnon is an elementary excitation of magnetic materials. It is a quantum of a spin wave that is a collective motion of the order parameter, which is the magnetization \vec{M} for a ferromagnet.

For example, consider a one-dimensional Heisenberg ferromagnet which consists of N spin- $\frac{1}{2}$ atoms. Under the periodic boundary condition, its Hamiltonian is given by

$$\hat{H} = -J \sum_{l=1}^N \hat{\sigma}(l) \cdot \hat{\sigma}(l+1). \quad (1)$$

Here, J is a positive constant, and $\hat{\sigma}(l) \equiv (\hat{\sigma}_x(l), \hat{\sigma}_y(l), \hat{\sigma}_z(l))$, where $\hat{\sigma}_x(l), \hat{\sigma}_y(l), \hat{\sigma}_z(l)$ are Pauli operators on site l . We denote eigenvectors of $\hat{\sigma}_z$ corresponding to eigenvalues $+1$ and -1 by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. One of the ground states of the Hamiltonian is $|\downarrow^{\otimes N}\rangle$, in which \vec{M} points to the $-z$ direction. The state in which one magnon with wave number k is excited on this ground state is

$$|\psi_{k;N}\rangle \equiv \frac{1}{\sqrt{N}} \sum_{l=1}^N e^{ikl} \hat{\sigma}_+(l) |\downarrow^{\otimes N}\rangle, \quad (2)$$

where $\hat{\sigma}_+(l) \equiv [\hat{\sigma}_x(l) + i\hat{\sigma}_y(l)]/2$. The excitation energy of $|\psi_{k;N}\rangle$ is easily calculated as

$$E_{k;N} = 8J \sin^2 \frac{k}{2}. \quad (3)$$

It goes to zero as $k \rightarrow 0$ because a magnon is a Nambu-Goldstone boson. The dispersion relation for small k is non-

linear because a magnon is a nonrelativistic excitation. Because of the periodic boundary condition, k takes discrete values in the first Brillouin zone, $-\pi < k \leq \pi$;

$$k = \frac{2\pi}{N} j \left(j : \text{integer}, -\frac{N}{2} < j \leq \frac{N}{2} \right). \quad (4)$$

It is conventional to define the ‘‘creation operator’’ of a magnon with wave number k by

$$\hat{M}_k^\dagger \equiv \frac{1}{\sqrt{N}} \sum_{l=1}^N e^{ikl} \hat{\sigma}_+(l). \quad (5)$$

The commutation relations are calculated as

$$[\hat{M}_k^\dagger, \hat{M}_{k'}^\dagger] = [\hat{M}_k, \hat{M}_{k'}] = 0, \quad (6)$$

$$[\hat{M}_k, \hat{M}_{k'}^\dagger] = -\frac{1}{N} \sum_{l=1}^N e^{i(k'-k)l} \hat{\sigma}_z(l). \quad (7)$$

When the number m of magnons is much smaller than N , Eq. (7) can be approximated as

$$[\hat{M}_k, \hat{M}_{k'}^\dagger] \simeq \frac{1}{N} \sum_{l=1}^N e^{i(k'-k)l} = \delta_{k,k'}. \quad (8)$$

Therefore magnons behave as bosons when $m \ll N$.

Using the creation operators, we define the m -magnon state with wave numbers k_1, k_2, \dots, k_m by

$$|\psi_{k_1, k_2, \dots, k_m; N}\rangle \equiv \frac{G_{k_1, k_2, \dots, k_m; N}}{\sqrt{n_a! n_b! \dots}} \prod_{i=1}^m \hat{M}_{k_i}^\dagger |\downarrow^{\otimes N}\rangle. \quad (9)$$

Here, $1/\sqrt{n_a! n_b! \dots}$ is the usual normalization factor for bosons, where n_ν ($\nu = a, b, \dots$) denotes the number of k_i 's having equal values, and $G_{k_1, \dots, k_m; N}$ is an extra normalization factor which comes from the fact that magnons are not strictly bosons. Without loss of generality, we henceforth assume that

$$k_1 \leq k_2 \leq \dots \leq k_m. \quad (10)$$

When $m \ll N$, the magnons behave as bosons so that $G_{k_1, \dots, k_m; N} = 1$ and

$$\langle \psi_{k_1, k_2, \dots, k_m; N} | \psi_{k'_1, k'_2, \dots, k'_m; N} \rangle = \delta_{k_1, k'_1} \delta_{k_2, k'_2} \dots \delta_{k_m, k'_m} \quad (11)$$

to a good approximation. On the other hand, the deviations from these relations become significant when $m = O(N)$.

An m -magnon state with a small density ($m/N \ll 1$) of magnons is an approximate energy eigenstate. Although an m -magnon state with a large number [$m = O(N)$] of magnons is not generally a good approximation to energy eigenstate, such a state is frequently used in discussions on a macroscopic order because many magnetic phase transitions can be regarded as condensation of $O(N)$ magnons. For example, the state in which \vec{M} points to a direction with the direction vector $(\sin\theta \cos\alpha, \sin\theta \sin\alpha, \cos\theta)$ can be described as

$$|(\theta\alpha)^{\otimes N}\rangle = \left(e^{-i\alpha} \cos \frac{\theta}{2} |\uparrow\rangle + \sin \frac{\theta}{2} |\downarrow\rangle \right)^{\otimes N} \quad (12)$$

$$= \sum_{m=0}^N e^{-im\alpha} \sqrt{B_m} |\psi_{(k=0)^m, N}\rangle, \quad (13)$$

where $|\psi_{(k=0)^m, N}\rangle$ is the m -magnon state with $k_1 = \dots = k_m = 0$, and B_m is the binomial coefficient;

$$B_m \equiv \binom{N}{m} \left(\cos^2 \frac{\theta}{2} \right)^m \left(\sin^2 \frac{\theta}{2} \right)^{N-m}. \quad (14)$$

When $\theta \neq \pi$, B_m has a peak at $m = N \cos^2(\theta/2) = O(N)$, and thus a macroscopic number of magnons are ‘‘condensed.’’

Note that $|\downarrow^{\otimes N}\rangle$ and $|(\theta\alpha)^{\otimes N}\rangle$ belong to the *same* Hilbert space because we assume that N is large but *finite*, although they will belong to different Hilbert spaces if we let $N \rightarrow \infty$. For the same reason, all $|\psi_{k_1, k_2, \dots, k_m, N}\rangle$'s belong to the same Hilbert space even if $m = O(N)$.

III. INDEX OF MACROSCOPIC ENTANGLEMENT

In this section, we present the index of macroscopic entanglement, and an efficient method of computing it. We also explain its physical meanings by giving a few examples. The relation between this index and stabilities of many-body states will be explained in Sec. VI.

A. Index p

We are most interested in superposition of macroscopically distinct states, which has been attracting much attention for many years [24–28]. We note that such superposition was defined rather ambiguously. For example, the ‘‘disconnectivity’’ defined in Ref. [24] is not invariant under changes of canonical variables, such as from the pairs of positions and momenta to the pair of a field and its canonical conjugate. Furthermore, in much literature the energy scale is not specified to determine the degrees of freedom involved in the superposition. However, the degrees of freedom, hence the disconnectivity, usually grows (decreases) quickly with increasing (decreasing) the energy scale under consideration [29]. On the other hand, SM proposed a different definition that is free from these ambiguities. We therefore follow SM.

We first fix the energy range under consideration. For definiteness we assume that in that energy range the system can be regarded as N spin- $\frac{1}{2}$ atoms, which are arranged on a one-dimensional lattice. We note that two states are ‘‘macroscopically distinct’’ iff some macroscopic variable(s) takes distinct values for those states. As a macroscopic variable, it is natural to consider the sum or average of local observables over a macroscopic region [30]. Since the average can be directly obtained from the sum, we only consider the sum in the following. That is, we consider additive operators [31], which take the following form: $\hat{A} = \sum_{l=1}^N \hat{a}(l)$. Here, $\hat{a}(l)$ is a local operator on site l , which, for the spin system under consideration is a linear combination of the Pauli operators $\hat{\sigma}_x(l)$, $\hat{\sigma}_y(l)$, $\hat{\sigma}_z(l)$ and the identity operator $\hat{1}(l)$ on site l .

Since we will consider all possible additive operators, we do *not* assume that $\hat{a}(l')$ ($l' \neq l$) is a spatial translation of $\hat{a}(l)$.

Two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, are macroscopically distinct iff some additive observable(s) \hat{A} takes macroscopically distinct values for those states in the sense that

$$\langle \psi_1 | \hat{A} | \psi_1 \rangle - \langle \psi_2 | \hat{A} | \psi_2 \rangle = O(N). \quad (15)$$

Therefore if a pure state $|\psi\rangle$ has fluctuation of this order of magnitude, i.e., if $\delta A \equiv [\langle \psi | \Delta \hat{A}^\dagger \Delta \hat{A} | \psi \rangle]^{1/2} = O(N)$ for some additive observable(s) \hat{A} , where $\Delta \hat{A} \equiv \hat{A} - \langle \psi | \hat{A} | \psi \rangle$, then the state is a superposition of macroscopically distinct states. On the other hand, if $\delta A = o(N)$ [23] for every additive observable \hat{A} the state has macroscopically definite values for all additive observables. A typical magnitude of δA for such a state is $\delta A = O(N^{1/2})$ [32]. To express these ideas in a simple form, we define an index p for an arbitrary pure state $|\psi\rangle$ by the asymptotic behavior (for large N) of fluctuation of the additive observable that exhibits the largest fluctuation for that state [33]:

$$\sup_{\hat{A} \in \mathcal{A}} \langle \psi | \Delta \hat{A}^\dagger \Delta \hat{A} | \psi \rangle = O(N^p). \quad (16)$$

Here, \mathcal{A} is the set of all additive operators. According to the above argument, $|\psi\rangle$ is a superposition of macroscopically distinct states iff $p = 2$, and for pure states p is the essentially unique index that characterizes such a superposition. We therefore say that a pure state is *macroscopically entangled* iff $p = 2$.

B. An efficient method of computing p

It is well known that many entanglement measures which are defined as an extremum are intractable [11,13–15]. In contrast, there is an efficient method of computing the index p [3]. We here explain the method briefly assuming an N spin- $\frac{1}{2}$ system.

Any local operator $\hat{a}(l)$ on site l can be expressed as a linear combination of $\hat{\sigma}_x(l)$, $\hat{\sigma}_y(l)$, $\hat{\sigma}_z(l)$, and $\hat{1}(l)$. Since the identity operator $\hat{1}$ does not have fluctuation for any state, we can limit ourselves to local operators that are linear combinations of Pauli operators. Therefore an additive observable in question generally takes the following form:

$$\hat{A} = \sum_{l=1}^N \hat{a}(l) = \sum_{l=1}^N \sum_{\alpha=x,y,z} c_{\alpha l} \hat{\sigma}_\alpha(l), \quad (17)$$

where $c_{\alpha l}$'s are complex coefficients (see Sec. IV E). Since the local operators should not depend on N (because otherwise \hat{A} would not become additive), $c_{\alpha l}$'s should not depend on N , hence the sum $\sum_l \sum_{\alpha} |c_{\alpha l}|^2$ is $O(N)$. Since we are interested only in the power p of $\langle \psi | \Delta \hat{A}^\dagger \Delta \hat{A} | \psi \rangle = O(N^p)$, we can normalize $c_{\alpha l}$ without loss of generality as

$$\sum_{l=1}^N \sum_{\alpha=x,y,z} |c_{\alpha l}|^2 = N. \quad (18)$$

The fluctuation of \hat{A} for a given state $|\psi\rangle$ is expressed as

$$\langle \psi | \Delta \hat{A}^\dagger \Delta \hat{A} | \psi \rangle = \sum_{\alpha, l, \beta, l'} c_{\alpha l}^* c_{\beta l'} V_{\alpha l, \beta l'}, \quad (19)$$

where $V_{\alpha l, \beta l'}$ is the Hermitian matrix defined by

$$V_{\alpha l, \beta l'} \equiv \langle \psi | \Delta \hat{\sigma}_\alpha(l) \Delta \hat{\sigma}_\beta(l') | \psi \rangle, \quad (20)$$

which we call the *variance-covariance matrix* (VCM) for $|\psi\rangle$. It is seen from Eq. (19) that eigenvalues of this matrix are non-negative, and that $\langle \psi | \Delta \hat{A}^\dagger \Delta \hat{A} | \psi \rangle$ takes the maximum value when $c_{\alpha l}$ is an eigenvector of the VCM corresponding to the maximum eigenvalue e_{\max} . By taking $c_{\alpha l}$ of Eq. (19) as such an eigenvector, we obtain

$$\sup_{\hat{A} \in \mathcal{A}} \langle \psi | \Delta \hat{A}^\dagger \Delta \hat{A} | \psi \rangle = e_{\max} N. \quad (21)$$

Therefore e_{\max} is related to p as

$$e_{\max} = O(N^{p-1}). \quad (22)$$

For example, $p=1$ if $e_{\max}=O(1)$ whereas $p=2$ if $e_{\max}=O(N)$.

Note that we can evaluate p of a given state using this method in a polynomial time of the number N of spins, because we have only to calculate the maximum eigenvalue of a VCM, which is a $3N \times 3N$ matrix.

C. Examples of macroscopically entangled states

The N -spin Greenberger-Horne-Zeilinger (GHZ) state, or the ‘‘cat’’ state, $|\text{GHZ}\rangle = 1/\sqrt{2}(|\downarrow^{\otimes N}\rangle + |\uparrow^{\otimes N}\rangle)$, violates a generalized Bell’s inequality by a macroscopic factor [4]. The index p correctly indicates that this state is macroscopically entangled, $p=2$ [2]. In contrast, $S_{N/2}(N)$ [which is defined by Eq. (30) below] of this state is extremely small; $S_{N/2}(N)=1$. It may be intuitively trivial that this state is macroscopically entangled. However, intuition is useless for more general states such as the following examples. The greatest advantage of using p is that it correctly judges the presence or absence of macroscopic entanglement for any complicated pure states.

For example, it was recently shown [2] that the quantum state of many qubits in a quantum computer performing Shor’s factoring algorithm is transformed in such a way that p is increased as the computation proceeds, and the state just after the modular exponentiation processes,

$$|\text{ME}\rangle \equiv \frac{1}{\sqrt{2^{N_1}}} \sum_{a=0}^{2^{N_1}-1} |a\rangle_1 |x^a \bmod M\rangle_2, \quad (23)$$

is a macroscopically entangled state. Here, $|\cdots\rangle_1 (|\cdots\rangle_2)$ represents a state of the first (second) register, $N_1 (2 \log M \leq N_1 < 2 \log M + 1)$ denotes the number of qubits in the first register, x is a randomly taken integer, and M is a large integer to be factored. This state was shown to play an essential role in Shor’s factoring algorithm [2]. Although *presence* of entanglement in this state is obvious, the *presence* of macroscopic entanglement was not revealed until an additive operator whose fluctuation is $O(N^2)$ was found in Ref. [2].

Another example is entanglement of ground states of antiferromagnets, which has recently been studied by many au-

thors using various measures [6–10]. It is well known that the *exact* ground state $|G_{\text{AF}}\rangle$ of the Heisenberg antiferromagnet on a two-dimensional square lattice of a finite size is *not* the Néel state but the *symmetric* state that possesses all the symmetries of the Hamiltonian [34]. We here point out that $|G_{\text{AF}}\rangle$ is entangled macroscopically $p=2$. In fact, the ground state has a long-range order [35],

$$\langle G_{\text{AF}} | (\hat{M}_\alpha^{\text{st}})^2 | G_{\text{AF}} \rangle \sim 0.117N^2 + 1.02N^{4/3}, \quad (24)$$

where $\hat{M}_\alpha^{\text{st}} \equiv \sum_{l=1}^N (-1)^l \hat{\sigma}_\alpha(l)$ is the staggered magnetization ($\alpha=x, y, z$). On the other hand, $\langle G_{\text{AF}} | \hat{M}_\alpha^{\text{st}} | G_{\text{AF}} \rangle = 0$ by symmetry. Therefore the order parameter $\hat{M}_\alpha^{\text{st}}$ of the antiferromagnetic phase transition exhibits a huge fluctuation,

$$\langle G_{\text{AF}} | (\Delta \hat{M}_\alpha^{\text{st}})^2 | G_{\text{AF}} \rangle = O(N^2). \quad (25)$$

This shows that $p=2$ for $|G_{\text{AF}}\rangle$. Note that such a macroscopically entangled ground state appears generally in a *finite* system that will exhibit a phase transition as $N \rightarrow \infty$ if the order parameter does not commute with the Hamiltonian [1,36]. For example, the ground state of interacting bosons [37,38], for which the order parameter is the field operator of the bosons, is entangled macroscopically. Moreover, the ground state of the transverse Ising model, whose entanglement has recently been studied using various measures [8,11–14], also has $p=2$ when the transverse magnetic field is below the critical point.

As demonstrated by these examples, the index p captures the presence or absence of certain anomalous features, which are sometimes hard to find intuitively, of pure quantum states in finite macroscopic systems. Furthermore, as we will explain in Sec. VI, p is directly related to fundamental stabilities of many-body states.

IV. MACROSCOPIC ENTANGLEMENT OF m -MAGNON STATES

We now study macroscopic entanglement of magnon states (9) with various densities and wave numbers by evaluating the index p .

A. States to be studied

Most relevant parameters characterizing the magnon states are the number of magnons, m , and the wave numbers of magnons. Because of the Z_2 symmetry, we assume that $1 \leq m \leq N/2$ without loss of generality. Furthermore, we assume that N is even in order to avoid uninteresting complications.

Since we are interested in the asymptotic behavior for large N , only the order of magnitudes of these parameters is important. We therefore consider the following three cases [39]:

(a) $m=O(1)$.

(b) $m=O(N)$. All magnons have different wave numbers from each other, continuously occupying a part of the first Brillouin zone. Because of the translational invariance of the system in the k space, it is sufficient to calculate the case where the magnons continuously occupy the first Brillouin

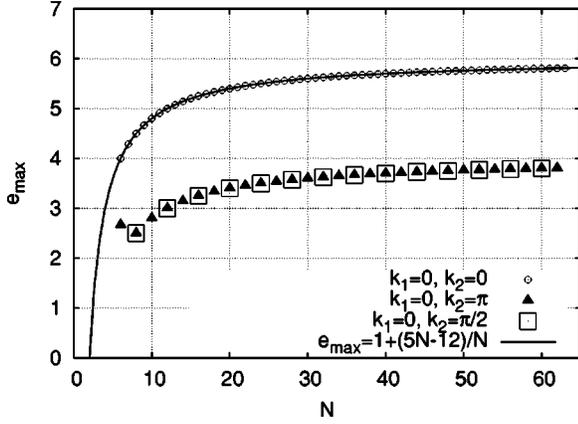


FIG. 1. The maximum eigenvalue e_{\max} of the VCM of two-magnon states with wave numbers k_1 and k_2 as functions of the number N of spins. Because of the translational invariance of the system in the k space, we take $k_1=0$ without loss of generality. The solid line represents the analytic expression $e_{\max}=1+(5N-12)/N$, Eq. (27), which assumes that all wave numbers are equal.

zone from the bottom, i.e., their wave numbers are $0, \pm 2\pi/N, \pm 4\pi/N, \dots$, respectively.

(c) $m=O(N)$. All magnons have equal wave numbers k . Because of the translational invariance of the system in the k space, we can take $k=0$ without loss of generality.

Furthermore, a small number [$=O(1)$] of magnons with arbitrary wave numbers may be added to these states. It is expected and will be confirmed in the following that the addition does not alter the value of p .

B. Case (a)

In Fig. 1 we plot numerical results for e_{\max} of two-magnon states as functions of N . The result for $k_1=k_2$ can also be obtained from the analytic expression, Eq. (27). These results show that excitation of a small number [$O(1)$] of magnons on the ferromagnetic ground state $|\downarrow^{\otimes N}\rangle$, which is a separable state, does not change the value of p . It is thus concluded that magnon states for case (a) are *not* macroscopically entangled.

C. Case (b)

To investigate p for case (b), we evaluate e_{\max} for various magnon densities assuming that all magnons have different wave numbers from each other, continuously occupying the first Brillouin zone from the bottom. The results are plotted in Fig. 2 for $m=N/2$, $N/4$, and $N/6$ as functions of N . It is seen that $e_{\max} \sim \text{constant}$, hence $p=1$. We also confirmed (not shown in the figure) that addition of small number of magnons with arbitrary wave numbers does not alter the value of p . We thus conclude that magnon states for case (b) are *not* macroscopically entangled.

D. Case (c)

If the wave numbers of all magnons are equal, we can calculate e_{\max} analytically as follows. Since we can take k

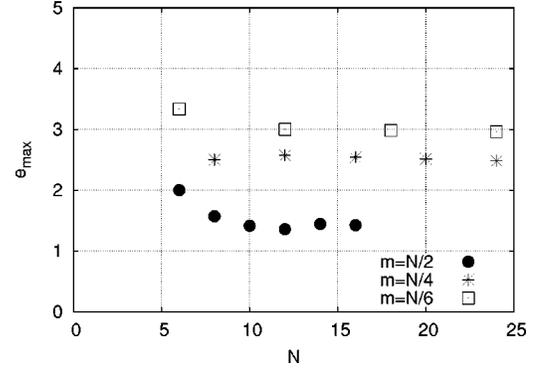


FIG. 2. The maximum eigenvalue e_{\max} of the VCM of m -magnon states with $m=N/2$, $N/4$, and $N/6$ as functions of the number N of spins. The wave numbers of magnons are all different taking the values $0, \pm 2\pi/N, \pm 4\pi/N, \dots$, respectively, i.e., the first Brillouin zone is continuously occupied from the bottom.

$=0$ by symmetry, we calculate the VCM of $|\psi_{(k=0)^m;N}\rangle$. From calculations described in the Appendix, we obtain the VCM and the maximum eigenvalue as

$$V_{j,j'} = \begin{cases} 1 & (j=j', 1 \leq j \leq 2N) \\ 1 - W_3^2 & (j=j', 2N+1 \leq j \leq 3N) \\ W_1 & (j \neq j', 1 \leq j, j' \leq N) \\ W_1 & (j \neq j', N+1 \leq j, j' \leq 2N) \\ W_2 - W_3^2 & (j \neq j', 2N+1 \leq j, j' \leq 3N) \\ -iW_3 & (j=j' - N, N+1 \leq j' \leq 2N) \\ iW_3 & (j=j' + N, 1 \leq j' \leq N) \\ 0 & (\text{others}), \end{cases} \quad (26)$$

$$e_{\max} = 1 + (N-1)W_1 + W_3 = 1 + \frac{2mN - 2m^2 + N - 2m}{N}, \quad (27)$$

where W_1 , W_2 , and W_3 are defined by Eqs. (A2), (A3), and (A1), respectively. We therefore find that $e_{\max}=O(N)$ for $m=O(N)$, hence $p=2$.

The solid line in Fig. 3 represents the analytic expression for e_{\max} , Eq. (27), for $m=N/2$. We also plot numerical results for the cases where the wave numbers of one or two magnons are different. It is seen that e_{\max} becomes smaller in the latter cases, as we have seen a similar tendency in Fig. 1. However, $e_{\max}=O(N)$ and thus $p=2$ in all three cases in Fig. 3. We therefore conclude that magnon states for case (c) are macroscopically entangled.

E. Additive operator with the maximum fluctuation

For a given state $|\psi\rangle$, we can obtain the additive operator \hat{A}_{\max} that has the maximum fluctuation $(\delta A_{\max})^2 \equiv \langle \psi | \Delta \hat{A}_{\max}^\dagger \Delta \hat{A}_{\max} | \psi \rangle = N e_{\max}$ for that state by inserting the eigenvector of the VCM belonging to the maximum eigenvalue e_{\max} into Eq. (17). However, \hat{A}_{\max} is generally non-

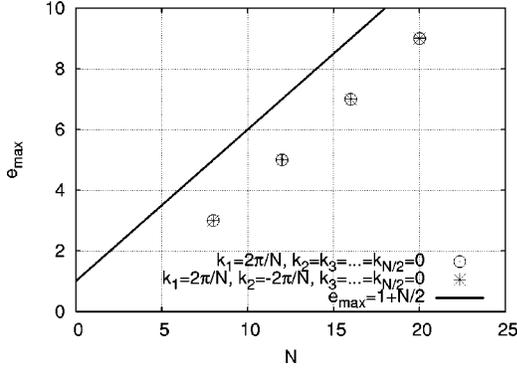


FIG. 3. The maximum eigenvalue e_{\max} of the VCM of $N/2$ -magnon states as functions of the number N of spins. Most magnons have equal wave numbers, i.e., most magnons are “condensed.” The solid line represents the analytic expression $e_{\max} = 1 + N/2$, Eq. (27), which assumes that all wave numbers are equal. The circle and cross represent numerical results for the cases where the wave numbers of one or two magnons are different. Because of the translational invariance in the k space, we take the wave number of the condensed magnons as 0 without loss of generality.

Hermitian because the eigenvector is generally complex. A non-Hermitian operator \hat{A} can be decomposed into the sum of two Hermitian operators \hat{A}' and \hat{A}'' as $\hat{A} = \hat{A}' + i\hat{A}''$. If \hat{A}' and \hat{A}'' commute with each other, they can be measured simultaneously with vanishing errors. Since the values of \hat{A} have one to one correspondence to the pairs of the values of \hat{A}' and \hat{A}'' , one can measure \hat{A} by simultaneously measuring \hat{A}' and \hat{A}'' if $[\hat{A}', \hat{A}''] = 0$. Note that in macroscopic systems $[\hat{A}', \hat{A}''] \approx 0$ to a good approximation for *any* additive operators \hat{A}' and \hat{A}'' because $[(\hat{A}'/N), (\hat{A}''/N)]$ is at most $O(1/N) \approx 0$. Therefore in macroscopic systems non-Hermitian additive operators can be measured to a good accuracy. Hence \hat{A}_{\max} can be measured even if it is non-Hermitian. One can also construct the Hermitian additive operators $\hat{A}'_{\max} \equiv (\hat{A}_{\max} + \hat{A}_{\max}^\dagger)/2$ and $\hat{A}''_{\max} \equiv (\hat{A}_{\max} - \hat{A}_{\max}^\dagger)/2i$, which are the “real” and “imaginary” parts, respectively, of \hat{A}_{\max} . Using the triangle inequality $\|\Delta \hat{A}_{\max} |\psi\rangle\| \leq \|\Delta \hat{A}'_{\max} |\psi\rangle\| + \|\Delta \hat{A}''_{\max} |\psi\rangle\|$, we can easily show that either (or both) of $\delta A'_{\max}$ or $\delta A''_{\max}$ is of the same order as δA_{\max} .

For $|\psi_{(k=0)^m; N}\rangle$ with $m = O(N)$, for example, the eigenvector belonging to the maximum eigenvalue (27) is

$$\frac{1}{2} \left(\overbrace{1, \dots, 1}^N, \overbrace{i, \dots, i}^N, \overbrace{0, \dots, 0}^N \right)^t, \quad (28)$$

which gives the maximally fluctuating additive operator as

$$\hat{A}_{\max} = \frac{1}{2} \sum_{l=1}^N [1 \cdot \hat{\sigma}_x(l) + i \cdot \hat{\sigma}_y(l) + 0 \cdot \hat{\sigma}_z(l)] = \sum_{l=1}^N \hat{\sigma}_+(l), \quad (29)$$

for which $(\delta A_{\max})^2 = O(N^2)$. Although this operator is not Hermitian, it can be measured to a good accuracy if $N \gg 1$.

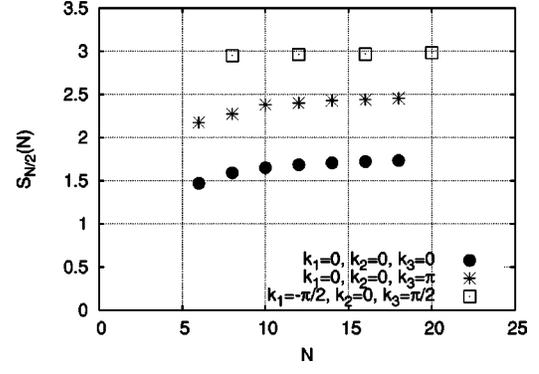


FIG. 4. The von Neumann entropy of a subsystem, $S_{N/2}(N)$, of three-magnon states with wave numbers k_1, k_2 , and k_3 as functions of the number N of spins. Because of the translational invariance of the system in the k space, we take $k_2 = 0$ without loss of generality.

Or, let us define Hermitian operators $\hat{A}'_{\max} \equiv (\hat{A}_{\max} + \hat{A}_{\max}^\dagger)/2 = \frac{1}{2} \sum_{l=1}^N \hat{\sigma}_x(l)$ and $\hat{A}''_{\max} \equiv (\hat{A}_{\max} - \hat{A}_{\max}^\dagger)/2i = \frac{1}{2} \sum_{l=1}^N \hat{\sigma}_y(l)$. Since $|\psi_{(k=0)^m; N}\rangle$ is symmetric under rotations about the z axis, we can show that $(\delta A'_{\max})^2 = (\delta A''_{\max})^2 = O(N^2)$ in this case. It is worth mentioning that \hat{A}'_{\max} corresponds to the eigenvector belonging to the second largest eigenvalue e_4 , which is given by Eq. (A4) and is of $O(N)$.

V. BIPARTITE ENTANGLEMENT OF m -MAGNON STATES

For a comparison purpose, we now calculate the degree of bipartite entanglement of magnon states that have been studied in the previous section. For a measure of bipartite entanglement, we use the von Neumann entropy of the reduced density operator of a subsystem. That is, we halve the N -spin system and evaluate the reduced density operator $\hat{\rho}_{N/2}(N)$ of one of the halves. The von Neumann entropy is defined by

$$S_{N/2}(N) \equiv -\text{Tr}[\hat{\rho}_{N/2}(N) \log_2 \hat{\rho}_{N/2}(N)], \quad (30)$$

which ranges from 0 to $N/2$. Although $S_{N/2}(N)$ for the case where all wave numbers are equal was discussed by Stockton *et al.* [15], we here evaluate $S_{N/2}(N)$ systematically for all the three cases listed in Sec., IV A.

A. Case (a)

To evaluate $S_{N/2}(N)$, we halve the N -spin system into two subsystems A and B . Accordingly, we decompose $|\psi_{k_1, k_2, \dots, k_m; N}\rangle$ into the sum of products of $|\psi_{k_1, k_2, \dots; N/2}\rangle$'s of A and B .

When all wave numbers are different from each other, an m -magnon state can be decomposed as

$$\begin{aligned}
 |\psi_{k_1, k_2, \dots, k_m; N}\rangle &= G_{k_1, k_2, \dots, k_m; N} \prod_{i=1}^m \hat{M}_{k_i}^\dagger | \downarrow^{\otimes N} \rangle = \frac{G_{k_1, k_2, \dots, k_m; N}}{\sqrt{2^m}} \left(G_{k_1, \dots, k_m; N/2}^{-1} | \downarrow^{\otimes N/2} \rangle |\psi_{k_1, \dots, k_m; N/2}\rangle + \sum_{i=1}^m e^{ik_i N/2} G_{k_i; N/2}^{-1} G_{k_1, \dots, \tilde{k}_i, \dots, k_m; N/2}^{-1} |\psi_{k_i; N/2}\rangle \right. \\
 &\quad \times |\psi_{k_1, \dots, \tilde{k}_i, \dots, k_m; N/2}\rangle + \sum_{i=1}^{m-1} \sum_{j=i+1}^m e^{ik_i N/2 + ik_j N/2} G_{k_i, k_j; N/2}^{-1} G_{k_1, \dots, \tilde{k}_i, \dots, \tilde{k}_j, \dots, k_m; N/2}^{-1} |\psi_{k_i, k_j; N/2}\rangle |\psi_{k_1, \dots, \tilde{k}_i, \dots, \tilde{k}_j, \dots, k_m; N/2}\rangle \\
 &\quad \left. + \dots + e^{ik_1 N/2 + \dots + ik_m N/2} G_{k_1, \dots, k_m; N/2}^{-1} |\psi_{k_1, \dots, k_m; N/2}\rangle | \downarrow^{\otimes N/2} \rangle \right), \quad (31)
 \end{aligned}$$

where $\tilde{}$ denotes absence, and the prefactor $1/\sqrt{2^m}$ comes from the prefactor $1/\sqrt{N}$ in Eq. (5). When $m=2$, for example,

$$\begin{aligned}
 |\psi_{k_1, k_2; N}\rangle &= \frac{G_{k_1, k_2; N}}{2} (G_{k_1, k_2; N/2}^{-1} | \downarrow^{\otimes N/2} \rangle |\psi_{k_1, k_2; N/2}\rangle \\
 &\quad + e^{ik_1 N/2} G_{k_1; N/2}^{-1} G_{k_2; N/2}^{-1} |\psi_{k_1; N/2}\rangle |\psi_{k_2; N/2}\rangle \\
 &\quad + e^{ik_2 N/2} G_{k_1; N/2}^{-1} G_{k_2; N/2}^{-1} |\psi_{k_2; N/2}\rangle |\psi_{k_1; N/2}\rangle \\
 &\quad + e^{ik_1 N/2 + ik_2 N/2} G_{k_1, k_2; N/2}^{-1} |\psi_{k_1, k_2; N/2}\rangle | \downarrow^{\otimes N/2} \rangle), \quad (32)
 \end{aligned}$$

which means that the state is the superposition of the following four ($=2^2$) states: (i) both magnons are in subsystem B , (ii) the magnon with k_1 is in A whereas the magnon with k_2 is in B , (iii) the magnon with k_2 is in A whereas the magnon with k_1 is in B , and (iv) both magnons are in A . As N is increased in decomposition (31) (while m is fixed), all G 's $\rightarrow 1$ and 2^m vectors on the right-hand side tend to become orthonormalized. This means that decomposition (31) becomes the Schmidt decomposition, in which the Schmidt rank is 2^m and all the Schmidt coefficients are equal (except for the phase factors). We thus obtain

$$\lim_{N \rightarrow \infty} S_{N/2}(N) = - \sum_{i=1}^{2^m} \left(\frac{1}{\sqrt{2^m}} \right)^2 \log_2 \left(\frac{1}{\sqrt{2^m}} \right)^2 = m, \quad (33)$$

(m :fixed)

i.e., $S_{N/2}(N) = O(1)$ for fixed m . Note that m is the maximum value of $S_{N/2}(N)$ among states whose Schmidt rank is 2^m .

When some of the wave numbers are equal, the Schmidt rank becomes smaller because magnons having equal wave numbers are indistinguishable. For example, if $k_1 = k_2 \equiv k$, the two-magnon state

$$|\psi_{k, k; N}\rangle = \frac{G_{k, k; N}}{\sqrt{2}} (\hat{M}_k^\dagger)^2 | \downarrow^{\otimes N} \rangle \quad (34)$$

is decomposed as

$$\begin{aligned}
 |\psi_{k, k; N}\rangle &= \frac{G_{k, k; N}}{2} (G_{k, k; N/2}^{-1} | \downarrow^{\otimes N/2} \rangle |\psi_{k, k; N/2}\rangle \\
 &\quad + \sqrt{2} e^{ikN/2} G_{k; N/2}^{-1} G_{k; N/2}^{-1} |\psi_{k; N/2}\rangle |\psi_{k; N/2}\rangle \\
 &\quad + e^{ikN} G_{k, k; N/2}^{-1} |\psi_{k, k; N/2}\rangle | \downarrow^{\otimes N/2} \rangle). \quad (35)
 \end{aligned}$$

In contrast to Eq. (32), this is the superposition of the following *three* ($<2^2$) states; (i) both magnons are in B , (ii) one magnon is in A and the other is in B , and (iii) both magnons are in A . The Schmidt rank is thus decreased. Furthermore, the Schmidt coefficients do not take the same value. As a result of these, $S_{N/2}(N)$ becomes smaller than Eq. (33);

$$\lim_{N \rightarrow \infty} S_{N/2}(N) < m, \quad (36)$$

(m :fixed)

from which we again have $S_{N/2}(N) = O(1)$.

Therefore we conclude that the bipartite entanglement of magnon states in case (a) is small in the sense that

$$S_{N/2}(N) = O(1). \quad (37)$$

As a demonstration, we plot numerical results for $S_{N/2}(N)$ as functions of N in Fig. 4, for three-magnon states in the following three cases; (i) three magnons have different wave numbers k_1, k_2, k_3 , (ii) two magnons have equal wave numbers $k_1 = k_2$ whereas one magnon has another wave number k_3 , and (iii) three magnons have equal wave numbers $k_1 = k_2 = k_3$. Formulas (33) and (36) are confirmed. Furthermore, it is seen that the departure of magnons from ideal bosons becomes significant for small N , and that $S_{N/2}(N)$ approaches the limiting values for $N \rightarrow \infty$ from below. This may be understood from the discussions of the following subsections.

Note that the result (37) agrees in some sense with the result of Sec. IV B, in which we have seen that the states are not macroscopically entangled. However, we will see in the following that such a simple agreement is not obtained in cases (b) and (c).

B. Case (b)

In case (b), the previous argument on $\lim_{N \rightarrow \infty} S_{N/2}(N)$ does not hold because the departure of magnons from ideal bosons is significant when $m = O(N)$. In fact, the vectors in Eq. (31) do not become orthonormalized as $N \rightarrow \infty$. Hence Eq. (31) does not become the Schmidt decomposition, and it can be further arranged until it becomes the Schmidt decomposition. Therefore we expect that the Schmidt rank is less than 2^m , and $\lim_{N \rightarrow \infty} S_{N/2}(N) < m$.

To see more details, we have calculated $S_{N/2}(N)$ numerically. The results are plotted as functions of N in Fig. 5 for case (b) with $m = N/2$, $N/4$, and $N/6$. It is found that the

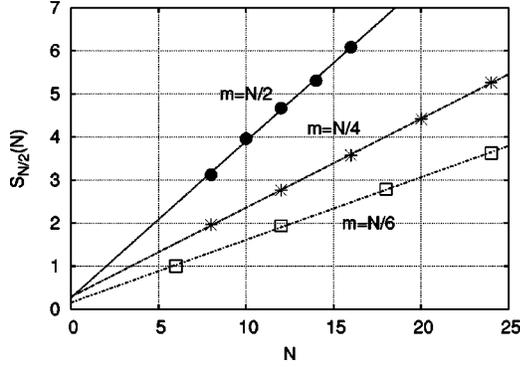


FIG. 5. The von Neumann entropy of a subsystem, $S_{N/2}(N)$, of m -magnon states with $m=N/2$, $N/4$, and $N/6$ as functions of the number N of spins. The wave numbers of magnons are all different taking the values $0, \pm 2\pi/N, \pm 4\pi/N, \dots$, respectively, i.e., the first Brillouin zone is continuously occupied from the bottom. The lines represent the regression lines calculated with the least squares.

results are well approximated by the straight lines,

$$S_{N/2}(N) = aN + b, \quad (38)$$

which are also displayed in Fig. 5. The parameters a and b are determined by the least squares, whose values are tabulated in Table I. Since $0 < a < m/N$, we find that $S_{N/2}(N)$ is less than, but of the same order of magnitude as, the maximum value $N/2$;

$$S_{N/2}(N) = O(N). \quad (39)$$

We thus conclude that the bipartite entanglement of magnon states in case (b) is extremely large. This should be contrasted with the result of Sec. IV C, according to which these states are *not* macroscopically entangled.

C. Case (c)

We finally consider $S_{N/2}(N)$ in case (c). When all wave numbers are equal to zero, the m -magnon state $|\psi_{(k=0)^m; N}\rangle$ becomes identical to the ‘‘Dicke state’’ that was discussed by Stockton *et al.* [15]. According to their result,

$$S_{N/2}(N) = O(\log N) \quad (40)$$

when $m=O(N)$. Because of the translational invariance, this result also holds for $|\psi_{(k)^m; N}\rangle$ with other values of k . Since $1 \leq O(\log N) \ll O(N)$, we find that $S_{N/2}(N)$ is slightly larger than that of case (a), but much smaller than that of case (b).

TABLE I. The parameters a and b , which are calculated with the least squares, of the regression line Eq. (38) for m -magnon states of Fig. 5.

m	a	Asymptotic standard error	b	Asymptotic standard error
$N/2$	0.36	± 0.009	0.27	± 0.113
$N/4$	0.21	± 0.002	0.29	± 0.027
$N/6$	0.15	± 0.003	0.15	± 0.044

We therefore conclude that the bipartite entanglement of magnon states for case (c) is small. This should be contrasted with the result of IV D, according to which these states are macroscopically entangled.

VI. STABILITIES AND ENTANGLEMENT

It may be expected that a quantum state with larger entanglement would be more unstable. This naive expectation is, however, quite ambiguous for many-body systems. First of all, the degree of entanglement depends drastically on the measure or index used to quantify the entanglement, as we have shown above. Furthermore, ‘‘stability’’ can be defined in many different ways for many-body states.

SM considered the following two kinds of stabilities [1]. One is the stability against weak perturbations from noises or environments: A pure state is said to be *fragile* if its decoherence rate behaves as $\sim KN^{1+\delta}$ when perturbations from the noises or environments are weak, where δ is a positive constant. Such a state is extremely unstable in the sense that its decoherence rate *per spin* increases as $\sim KN^\delta$ with increasing N , until it becomes extremely large for huge N however small is the coupling constant between the system and the noise or environment. SM showed that pure states with $p=1$ never become fragile in any noises or environments, whereas pure states with $p=2$ can become fragile, depending on the spectral intensities of the noise or environment variables. The other stability considered by SM is the stability against local measurements: A state is said to be *stable against local measurements* if an ideal (projective) measurement of any observable at a point l does not alter the result of measurement of any observable at a distant point l' for sufficiently large $|l-l'|$. SM showed that this stability is equivalent to the ‘‘cluster property,’’ which is closely related to p , if the cluster property for finite systems is properly defined. For example, a state is *unstable* against local measurements if $p=2$, whereas a homogeneous [40] state with $p=1$ is stable.

We have shown that $p=1$ for magnon states of case (b). Therefore these states never become fragile in any noises or environments, and they are stable against local measurements. Since we have also shown that the bipartite entanglement of these states is extremely large, we find that the bipartite entanglement is basically independent of these fundamental stabilities. The same conclusion was obtained for chaotic quantum systems by two of the authors [3]: They showed that $p=1$ for almost all energy eigenstates of macroscopic chaotic systems whereas their bipartite entanglement is nearly maximum.

We have also shown that $p=2$ for magnon states of case (c). Therefore these states can become fragile, depending on the spectral intensities of the noise or environment variables. Furthermore, these states are unstable against local measurements. Since we have seen that the bipartite entanglement of these states is small, we find again that the bipartite entanglement is basically independent of these fundamental stabilities. To understand the physics of this conclusion, the following simple example may be helpful. The W state,

$$|W\rangle = \frac{1}{\sqrt{N}} \sum_{l=1}^N |\downarrow^{\otimes(l-1)} \uparrow \downarrow^{\otimes(N-l)}\rangle, \quad (41)$$

has the *same* value of $S_{N/2}(N)$ as the N -spin GHZ state, i.e., $S_{N/2}(N)=1$. On the other hand, $p=1$ for $|W\rangle$ whereas $p=2$ for $|\text{GHZ}\rangle$. As a result, the decoherence rate of $|W\rangle$ *never* exceeds $O(N)$ by *any* weak classical noises, whereas the decoherence rate of $|\text{GHZ}\rangle$ becomes as large as $O(N^2)$ in a long-wavelength noise. Furthermore, $|W\rangle$ is *stable* against local measurements, whereas $|\text{GHZ}\rangle$ is unstable. These results are physically reasonable because the W state is nothing but a one-magnon state (with $k=0$), which can be generated easily by experiment: Such a state does not seem very unstable (see also Sec. VII B).

It should be mentioned that another stability was studied by Stockton *et al.* [15] for a special state of case (c), i.e., for a Dicke state which in our notation is written as $|\psi_{(k=0)^{N/2},N}\rangle$. They showed that a bipartite entanglement measure of $\hat{\rho}_{N-N'} \equiv \text{Tr}_{N'}(|\psi_{(k=0)^{N/2},N}\rangle\langle\psi_{(k=0)^{N/2},N}|)$ decreases very slowly as N' increases. Here, $\text{Tr}_{N'}$ means “trace out N' spins.” They thus concluded that the state is *robust*. Although their conclusion might look contradictory to our conclusion, there is no contradiction. The stability (robustness) as discussed by Stockton *et al.* is totally different from the fundamental stabilities that are discussed in the present paper. The Dicke state is “robust” in the sense of Stockton *et al.*, whereas in the senses of SM the state is “fragile” in noises or environments and “unstable” against local measurements. This demonstrates that stability can be defined in many different ways for many-body states.

VII. DISCUSSIONS

A. Relation to Bose-Einstein condensates

The “cluster property,” which is closely related to the index p , of condensed states of interacting many bosons was previously studied in Ref. [38]. It was shown there that $p=2$ for the ground state $|N, G\rangle$, which has a fixed number N of bosons, if N is large enough to give a finite density for a large volume. Since magnons are approximate bosons, magnon states of case (c) may be analogous to this state. Although deviations from ideal bosons become significant in case (c), the deviations may be partly regarded as effective interactions among magnons. This analogy intuitively explains our result that $p=2$ for magnon states of case (c).

It was also shown in Ref. [38] that $p=1$ for a generalized coherent state $|\alpha, G\rangle$, which was called there a coherent state of interacting bosons. This result may also be understood intuitively on the same analogy. That is, $|\alpha, G\rangle$ may be analogous to the state of Eq. (13), which has $p=1$ because it is a separable state as seen from Eq. (12). Therefore, by analogy, p should also be unity for $|\alpha, G\rangle$, in consistency with the result of Ref. [38], although $|\alpha, G\rangle$ is *not* separable.

Analogy like these may be useful for further understanding of systems of interacting many bosons and of many magnons.

B. What generates huge entanglement?

We have shown that states with huge entanglement, as measured by either p or $S_{N/2}(N)$, can be easily constructed by

simply exciting many magnons on a separable state. We now discuss the physical origin of this fact.

The most important point is that a magnon propagates spatially all over the magnet [41]. By the propagation, quantum coherence is established between spatially separated points [42]. Therefore by exciting a macroscopic number of magnons, one can easily construct states with huge entanglement.

Note that this should be common to most quantum systems, because Hamiltonians of most physical systems should have a term which causes spatial propagation. For example, such a term includes the nearest-neighbor interaction of spin systems, the kinetic-energy term of the Schrödinger equation of particles, the term composed of spatial derivative of a field operator in field theory. Therefore excitation of a macroscopic number of elementary excitations generates huge entanglement. Neither randomness nor elaborate tuning is necessary.

This observation will be useful for *theoretically* constructing states with huge entanglement. *Experimentally*, on the other hand, the stability should also be taken into account because unstable states would be hard to generate experimentally. We thus consider that states with $p=2$ should be much harder to generate experimentally than states with large $S_{N/2}(N)$. In other words, a state with large $S_{N/2}(N)$ would be able to be generated rather easily, e.g., by exciting many quasiparticles in a solid. In this respect, a naive expectation that states with large entanglement would be hard to generate experimentally is false: It depends on the measure or index that is used to quantify the entanglement.

ACKNOWLEDGMENTS

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APPENDIX: CALCULATION OF THE VCM AND ITS EIGENVALUES

The state vector of the m -magnon state with $k_1=\dots=k_m=0$ can be written as

$$\begin{aligned} & |\psi_{(k=0)^m,N}\rangle \\ &= \frac{1}{\sqrt{{}_N C_m}} \sum_{l_1} \sum_{l_2(>l_1)} \sum_{l_3(>l_2)} \dots \\ & \quad \times \sum_{l_m(>l_{m-1})} \hat{\sigma}_+(l_1)\hat{\sigma}_+(l_2)\dots\hat{\sigma}_+(l_m)|\downarrow^{\otimes N}\rangle, \end{aligned}$$

where

$${}_N C_m \equiv \binom{N}{m}.$$

Since the VCM is Hermitian, we have only to calculate the following correlations; $\langle\Delta\hat{\sigma}_x(l)\Delta\hat{\sigma}_x(l')\rangle$, $\langle\Delta\hat{\sigma}_x(l)\Delta\hat{\sigma}_y(l')\rangle$, $\langle\Delta\hat{\sigma}_x(l)\Delta\hat{\sigma}_z(l')\rangle$, $\langle\Delta\hat{\sigma}_y(l)\Delta\hat{\sigma}_y(l')\rangle$, $\langle\Delta\hat{\sigma}_y(l)\Delta\hat{\sigma}_z(l')\rangle$, $\langle\Delta\hat{\sigma}_z(l)\Delta\hat{\sigma}_z(l')\rangle$, where $\langle\cdot\rangle$ stands for $\langle\psi_{(k=0)^m,N}|\cdot|\psi_{(k=0)^m,N}\rangle$. Since $|\psi_{(k=0)^m,N}\rangle$ is an eigenvector of $\exp[-i\theta\sum_{l=1}^N\hat{\sigma}_z(l)]$, the

state vector is invariant under a rotation about the z axis. Therefore $\langle \Delta \hat{\sigma}_y(l) \Delta \hat{\sigma}_y(l') \rangle = \langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_x(l') \rangle$ and $\langle \Delta \hat{\sigma}_y(l) \Delta \hat{\sigma}_z(l') \rangle = \langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_z(l') \rangle$. Thus we calculate only

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_x(l') \rangle, \langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_y(l') \rangle,$$

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_z(l') \rangle, \langle \Delta \hat{\sigma}_z(l) \Delta \hat{\sigma}_z(l') \rangle.$$

We note that $\langle \hat{\sigma}_x(l) \rangle = \langle \hat{\sigma}_y(l) \rangle = 0$ by symmetry, and

$$\begin{aligned} \langle \hat{\sigma}_z(l) \rangle &= \frac{1}{N C_m} ({}_{N-1}C_{m-1} - {}_{N-1}C_m) \\ &= -\frac{N-2m}{N} \\ &\equiv -W_3. \end{aligned} \quad (\text{A1})$$

When $l=l'$, we easily obtain

$$\langle \hat{\sigma}_x(l) \hat{\sigma}_x(l) \rangle = \langle \hat{\sigma}_z(l) \hat{\sigma}_z(l) \rangle = 1, \langle \hat{\sigma}_x(l) \hat{\sigma}_z(l) \rangle = 0,$$

and

$$\langle \hat{\sigma}_x(l) \hat{\sigma}_y(l) \rangle = i \langle \hat{\sigma}_z(l) \rangle = -iW_3.$$

Therefore

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_x(l) \rangle = 1,$$

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_y(l) \rangle = -iW_3,$$

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_z(l) \rangle = 0,$$

$$\langle \Delta \hat{\sigma}_z(l) \Delta \hat{\sigma}_z(l) \rangle = 1 - W_3^2.$$

When $l \neq l'$, we note that

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_x(l') | \dots \downarrow \dots \uparrow \dots \rangle = 1,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_x(l') | \dots \uparrow \dots \downarrow \dots \rangle = 1,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_x(l') | \dots \downarrow \dots \downarrow \dots \rangle = 0,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_x(l') | \dots \uparrow \dots \uparrow \dots \rangle = 0,$$

where $|\dots \downarrow \dots \uparrow \dots \rangle$ is a state vector in which m spins including l' th spin are up, whereas $N-m$ spins including l th spin are down. We thus obtain

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_x(l') \rangle = \frac{{}_{N-2}C_{m-1}}{{}_N C_m} = \frac{2m(N-m)}{N(N-1)} \equiv W_1. \quad (\text{A2})$$

Furthermore, since

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_y(l') | \dots \downarrow \dots \uparrow \dots \rangle = i,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_y(l') | \dots \uparrow \dots \downarrow \dots \rangle = -i,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_y(l') | \dots \downarrow \dots \downarrow \dots \rangle = 0,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_x(l) \hat{\sigma}_y(l') | \dots \uparrow \dots \uparrow \dots \rangle = 0,$$

we obtain

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_y(l') \rangle = \frac{1}{{}_N C_m} (i {}_{N-2}C_{m-1} - i {}_{N-2}C_{m-1}) = 0.$$

It is obvious that $\langle \hat{\sigma}_x(l) \hat{\sigma}_z(l') \rangle = 0$ because $|\psi_{(k=0)^m; N}\rangle$ is a linear combination of vectors whose m spins are up and $N-m$ spins are down. Therefore

$$\langle \Delta \hat{\sigma}_x(l) \Delta \hat{\sigma}_z(l') \rangle = 0,$$

Finally, since

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_z(l) \hat{\sigma}_z(l') | \dots \downarrow \dots \uparrow \dots \rangle = -1,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_z(l) \hat{\sigma}_z(l') | \dots \uparrow \dots \downarrow \dots \rangle = -1,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_z(l) \hat{\sigma}_z(l') | \dots \downarrow \dots \downarrow \dots \rangle = 1,$$

$$\sqrt{{}_N C_m} \langle \psi_{(k=0)^m; N} | \hat{\sigma}_z(l) \hat{\sigma}_z(l') | \dots \uparrow \dots \uparrow \dots \rangle = 1,$$

we obtain

$$\begin{aligned} \langle \hat{\sigma}_z(l) \hat{\sigma}_z(l') \rangle &= \frac{1}{{}_N C_m} (-2 {}_{N-2}C_{m-1} + {}_{N-2}C_m + {}_{N-2}C_{m-2}) \\ &= \frac{N^2 - 4mN - N + 4m^2}{N(N-1)} \\ &\equiv W_2. \end{aligned} \quad (\text{A3})$$

Therefore

$$\langle \Delta \hat{\sigma}_z(l) \Delta \hat{\sigma}_z(l') \rangle = W_2 - W_3^2.$$

Combining these results, we obtain the VCM as Eq. (26), or

$$V = \begin{pmatrix} 1 & & W_1 & -iW_3 & 0 & & 0 \\ & \ddots & & & & \ddots & \\ W_1 & & 1 & 0 & -iW_3 & & 0 \\ iW_3 & & 0 & 1 & W_1 & & 0 \\ & \ddots & & & & \ddots & \\ 0 & & iW_3 & W_1 & 1 & & 0 \\ 0 & & & 0 & & & 1 - W_3^2 & W_2 - W_3^2 \\ & \ddots & & & & & & \ddots \\ & & & & 0 & & W_2 - W_3^2 & 1 - W_3^2 \end{pmatrix}.$$

We can calculate the eigenvalues e_j and the numbers M_j of the corresponding eigenvectors as

$$e_1 = 1 - W_2 = \frac{4m(N-m)}{N(N-1)}; \quad M_1 = N-1,$$

$$e_2 = 1 - W_3^2 + (N-1)(W_2 - W_3^2) = 0; \quad M_2 = 1,$$

$$e_3 = 1 + W_3 + (N-1)W_1 = \frac{2N-2m+2mN-2m^2}{N}; \quad M_3 = 1,$$

$$e_4 = 1 - W_3 + (N-1)W_1 = \frac{2m+2mN-2m^2}{N}; \quad M_4 = 1,$$

$$e_5 = 1 + W_3 - W_1 = \frac{2N^2 - 2N - 4mN + 2m + 2m^2}{N(N-1)};$$

$$M_5 = N-1,$$

$$e_6 = 1 - W_3 - W_1 = \frac{2m^2 - 2m}{N(N-1)}; \quad M_6 = N-1. \quad (\text{A4})$$

The largest one is e_3 , which degenerates with e_4 when $N=2m$. We thus obtain Eq. (27).

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- [29] For example, a ten-atom molecule can be regarded as a single particle, ten particles, and many more particles (nuclei and

electrons), in the energy ranges of $\sim\mu\text{eV}$, $\sim\text{meV}$, and $\sim\text{eV}$, respectively.

- [30] This statement would be understandable, from the discussion in endnote [31], for macroscopic variables that define equilibrium states. It is worth mentioning that the statement is also true for most macroscopic variables defining nonequilibrium states, such as the electric current density J . That is, the macroscopic current density J must be an average of the microscopic current density j over a macroscopic region. The spatial average introduces a smoothing effect, and J becomes a proper macroscopic variable.
- [31] Some of macroscopic variables, such as the volume and temperature, in thermodynamics cannot be represented as an additive operator. Although the volume is additive, it is usually considered as a boundary condition rather than a quantum-mechanical observable. We follow this convention. Regarding the temperature, it is a nonmechanical variable that can be defined only for equilibrium states. However, we note that at thermal equilibrium a macroscopic state is, hence the values of nonmechanical variables are, uniquely determined by a set of additive observables and the boundary conditions [L. Tisza, *Ann. Phys. (N.Y.)* **13**, 1 (1961); H. B. Callen, *Thermodynamics* (Wiley, New York, (1960)]. Therefore if two equilibrium states have distinct values of a nonmechanical variable they must have distinct values of some of the additive observables. It therefore seems that considering additive observables is sufficient.
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- [39] Although other states are possible, they are beyond the scope of the present paper.
- [40] For spatially homogeneous states, $p=1$ implies that the state has the cluster property. This is the case in the present paper because we only study translationally invariant states.
- [41] Although we have not considered time evolution in this paper, the spatial propagation has been incorporated into our analysis for the following reason. A magnon state created by the magnon operator \hat{M}_k^\dagger is an approximate eigenstate of a Hamiltonian of a magnet. Since an energy eigenstate is a stationary state, it is a state “after all possible propagation is finished.” Therefore spatial propagation has already been incorporated into magnon states.
- [42] The generation of entanglement by the Hamilton dynamics was suggested in many works, e.g., Ref. [24].