What is measured when fluctuation of macrovariables is measured ideally: Supplemental Material

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(Dated: February 9, 2016)

We first present generalization of measurement operators in Eq. (2). We then describe the experimental protocol for measuring the lhs's of Eqs. (4) and (7). Moreover, we show a version of the conditions and the statement of the quantum central limit theorem (QCLT), which is used for deriving the results of the paper. Furthermore, we present other theorems and their proofs to derive the main results of the paper. We finally present derivations of the main results of the paper, showing sufficient conditions on $f(x)$.

I. GENERALIZATION OF MEASUREMENT OPERATORS

We can generalize Eq. (2) as [1]

$$\hat{\rho}(a_\bullet) \propto (1/L) \sum_{\lambda=1}^{L} f^\lambda(\hat{a} - a_\bullet)\hat{\rho}_{\rm eq} f^\lambda(\hat{a} - a_\bullet) \tag{I.1}$$

with L different functions $f^1, f^2, \cdots, f^L$ that satisfy the conditions for f of the paper. As long as $L = O(1)$, one can obtain the results for $L > 1$ by averaging the results of the paper over all f^λ.

II. PROTOCOL FOR MEASURING THE LHS'S OF EQS. (4) AND (7)

The protocol for measuring the lhs's of Eqs. (4) and (7) is as follows.

One must perform many runs of experiments, which we label $r = 1, 2, \cdots$. In each experiment,

1) prepare the system in the same equilibrium state,
2) measure $\hat{a}$ at time t_r to get the outcome $a_{\bullet r}$,
3) measure $\hat{b}$ at time $t_r + t$ to get the outcome $b_{\bullet r}$.

By repeating 1)-3) many times, one gets a set of data $\{a_{\bullet r}, b_{\bullet r}\}_r$. Take arbitrarily a value $a_\bullet$, and select all data that satisfy $|a_{\bullet r} - a_\bullet| \ll 1$. By averaging $b_{\bullet r}$ over the selected data, one obtains $\langle \hat{b}(t) \rangle_{a_\bullet}$. By subtracting $\langle \hat{b} \rangle_{\rm eq}$, a way of getting which is obvious, one gets the lhs of Eq. (4). By multiplying this with $\Delta a_\bullet$, and by averaging over $a_\bullet$, one gets the lhs of Eq. (7).

III. THEOREMS

We write N for the size, such as the number of spins, of the quantum system. The thermodynamic limit (TDL) is briefly denoted by $\lim_{N\to\infty}$ throughout this Supplemental Material (and the paper). The justification of interchanging the order of the TDL and differentiation is one of the keys to the rigorous derivations of our results.

We let

$$\hat{A}_i = \sum_{\boldsymbol{r}} \hat{\alpha}_i(\boldsymbol{r}) \tag{III.1}$$

be an additive observable, where $\hat{\alpha}_i(\boldsymbol{r})$ is a local observable acting around $\boldsymbol{r}$, and $\boldsymbol{r}$ runs all over the volume of the system. The index $i = 1, 2, \ldots, M$ labels such additive observables of interest, where $M = O(1)$.

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A. Conditions for QCLT

The conditions for the QCLT are summarized as follows.

(C1) The following quantities are $O(1)$ (i.e., they have N-independent upper bounds):

$$\left\|[\hat{A}_i, \hat{\alpha}_j(\boldsymbol{r})]\right\|, \quad \left\|[\hat{A}_i, [\hat{A}_j, \hat{\alpha}_k(\boldsymbol{r})]\right\|. \tag{III.2}$$

(C2) The following quantity decays faster than $1/|\boldsymbol{r}-\boldsymbol{r}'|^{d+\varepsilon}$ ($\varepsilon > 0$ is some constant) with increasing the distance $|\boldsymbol{r}-\boldsymbol{r}'|$.

$$\langle \frac{1}{2}\{\Delta\hat{\alpha}_i(\boldsymbol{r}), \Delta\hat{\alpha}_j(\boldsymbol{r}')\}\rangle_{\rm eq} \tag{III.3}$$

(C3) For any simply connected subsystem Λ,

$$\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4\in\Lambda} |\langle \Delta\hat{\alpha}_i(\boldsymbol{r}_1)\Delta\hat{\alpha}_j(\boldsymbol{r}_2)\Delta\hat{\alpha}_k(\boldsymbol{r}_3)\Delta\hat{\alpha}_l(\boldsymbol{r}_4)\rangle| = O(|\Lambda|^2). \tag{III.4}$$

This implies, for example, that the forth moment of any additive operator is $O(N^2)$.

(C4) For some positive constant ε,

$$N^{1/2}c_N(N^{1/2d-\varepsilon}, N^{1/2d+\varepsilon}; \xi) \to 0 \quad (N\to\infty), \tag{III.5}$$

where

$$\begin{aligned} &c_N(r, N'; \xi) \\ &\equiv \sup\Big\{\Big|\langle e^{i\xi_1\hat{A}_1^{\Lambda+\Lambda'}/\sqrt{N}}\cdots e^{i\xi_M\hat{A}_M^{\Lambda+\Lambda'}/\sqrt{N}}\rangle_{\rm eq} - \langle e^{i\xi_1\hat{A}_1^{\Lambda}/\sqrt{N}}\cdots e^{i\xi_M\hat{A}_M^{\Lambda}/\sqrt{N}}\rangle_{\rm eq}\langle e^{i\xi_1\hat{A}_1^{\Lambda'}/\sqrt{N}}\cdots e^{i\xi_M\hat{A}_M^{\Lambda'}/\sqrt{N}}\rangle_{\rm eq}\Big|; \\ &\qquad d(\Lambda,\Lambda) \ge r, \min\{|\Lambda|, |\Lambda'|\} \le N'\Big\}. \end{aligned} \tag{III.6}$$

Here, Λ and Λ' are arbitrary simply-connected subsystems, and $\hat{A}_j^\Lambda$ denotes the restriction of $\hat{A}_j$ to Λ, i.e.,

$$\hat{A}_j^\Lambda = \sum_{\boldsymbol{r}\in\Lambda} \hat{\alpha}_j(\boldsymbol{r}). \tag{III.7}$$

B. Quantum Central Limit Theorems

Theorem 1 (QCLT for the characteristic function)**.** *If the conditions of Sec. III A are satisfied, then*

$$\lim_{N\to\infty} \langle \exp[i\frac{\xi_1}{\sqrt{N}}\Delta\hat{A}_1]\cdots\exp[i\frac{\xi_M}{\sqrt{N}}\Delta\hat{A}_M]\rangle_{\rm eq} = \exp[-\frac{1}{2}\sum_{j,k}\xi_j\xi_k s_{jk}], \tag{III.8}$$

where

$$s_{jk} \equiv \begin{cases} \lim_{N\to\infty}\frac{1}{N}\langle\Delta\hat{A}_j\Delta\hat{A}_k\rangle_{\rm eq} & (j\le k) \\ \lim_{N\to\infty}\frac{1}{N}\langle\Delta\hat{A}_k\Delta\hat{A}_j\rangle_{\rm eq} & (j>k) \end{cases} \tag{III.9}$$

$$= \begin{cases} \lim_{N\to\infty}\frac{1}{N}\langle\frac{1}{2}\{\Delta\hat{A}_j,\Delta\hat{A}_k\}\rangle_{\rm eq} + \lim_{N\to\infty}\frac{1}{N}\langle\frac{1}{2}[\Delta\hat{A}_j,\Delta\hat{A}_k]\rangle_{\rm eq} & (j\le k) \\ \lim_{N\to\infty}\frac{1}{N}\langle\frac{1}{2}\{\Delta\hat{A}_j,\Delta\hat{A}_k\}\rangle_{\rm eq} + \lim_{N\to\infty}\frac{1}{N}\langle\frac{1}{2}[\Delta\hat{A}_k,\Delta\hat{A}_j]\rangle_{\rm eq} & (j>k) \end{cases} \tag{III.10}$$

This theorem implies that the normalized fluctuation operators $\Delta\hat{A}_j/\sqrt{N}$ behave as c-numbers, i.e. as bosons, in the TDL. Indeed, one can consider that, in the TDL, the fluctuation operators $\Delta\hat{A}_j/\sqrt{N}$ form the CCR (Canonical Commutation Relation) algebra and the state is represented by a gaussian state for the algebra. This theorem is an extension to quantum mechanics of the (classical) central limit theorem, which states that the probabilistic distribution of an additive quantity tends to obey the gaussian distribution. For the detail and proof of the above theorem, see [2–4].

Actually, the conditions of Sec. III A are a bit stronger than required for the QCLT of this form. The conditions, though, are required for Theorem 3.

Furthermore, we can prove the following QCLTs for the first and second derivatives of the 'characteristic function,' which is a quantum analog of the characteristic function. These theorems will be used in Sec. IV.

Theorem 2 (QCLT for the first derivative)**.** *If the conditions of Sec. III A are satisfied, then*

$$\lim_{N\to\infty}\frac{\partial}{\partial\xi_j}\langle\exp[i\frac{\xi_1}{\sqrt{N}}\Delta\hat{A}_1]\cdots\exp[i\frac{\xi_M}{\sqrt{N}}\Delta\hat{A}_M]\rangle_{\rm eq}=\frac{\partial}{\partial\xi_j}\lim_{N\to\infty}\langle\exp[i\frac{\xi_1}{\sqrt{N}}\Delta\hat{A}_1]\cdots\exp[i\frac{\xi_M}{\sqrt{N}}\Delta\hat{A}_M]\rangle_{\rm eq} \tag{III.11}$$

$$=\frac{\partial}{\partial\xi_j}\exp\Big[-\frac{1}{2}\sum_{k,l}\xi_k\xi_l s_{kl}\Big]. \tag{III.12}$$

Theorem 3 (QCLT for the second derivative)**.** *If the conditions of Sec. III A are satisfied, then*

$$\lim_{N\to\infty}\frac{\partial^2}{\partial{\xi_j}^2}\langle\exp[i\frac{\xi_1}{\sqrt{N}}\Delta\hat{A}_1]\cdots\exp[i\frac{\xi_M}{\sqrt{N}}\Delta\hat{A}_M]\rangle_{\rm eq}=\frac{\partial^2}{\partial{\xi_j}^2}\lim_{N\to\infty}\langle\exp[i\frac{\xi_1}{\sqrt{N}}\Delta\hat{A}_1]\cdots\exp[i\frac{\xi_M}{\sqrt{N}}\Delta\hat{A}_M]\rangle_{\rm eq} \tag{III.13}$$

$$=\frac{\partial^2}{\partial{\xi_j}^2}\exp\Big[-\frac{1}{2}\sum_{k,l}\xi_k\xi_l s_{kl}\Big]. \tag{III.14}$$

To prove these theorems, we prepare some lemmas. Hereafter, we call an operator $\hat{U}(\boldsymbol{\xi})$ *a unitary operator generated by fluctuation* if

$$\hat{U}(\boldsymbol{\xi})=e^{i\xi_1\Delta\hat{A}_1/\sqrt{N}}\cdots e^{i\xi_L\Delta\hat{A}_L/\sqrt{N}}, \tag{III.15}$$

where $\hat{A}_1,\dots,\hat{A}_L$ are additive operators, and L is some constant of $O(1)$. We write $|\boldsymbol{\xi}|=\sum_j|\xi_j|$.

Lemma 1. *Assume that the system satisfies condition (C1). Let $\hat{U}(\boldsymbol{\xi})$ be a unitary operator generated by fluctuation, and $\hat{B}$ an additive operator. Then, there exists a positive constant K of $O(1)$ for which the following inequality holds for any $\boldsymbol{\xi}$:*

$$\frac{1}{\sqrt{N}}\left\|[\hat{U},\hat{B}]\right\|<K|\boldsymbol{\xi}|. \tag{III.16}$$

Proof. First, we prove the lemma for the case $\hat{U}(\xi)=e^{i\xi\Delta\hat{A}/\sqrt{N}}$.

$$\frac{1}{\sqrt{N}}\left\|[e^{i\xi\Delta\hat{A}/\sqrt{N}},\hat{B}]\right\|=\frac{1}{N}\left\|\int_0^\xi d\xi' ie^{i(\xi-\xi')\Delta\hat{A}/\sqrt{N}}[\hat{B},\hat{A}]e^{i\xi'\Delta\hat{A}/\sqrt{N}}\right\|$$
$$\le|\xi|\frac{1}{N}\left\|[\hat{A},\hat{B}]\right\|.$$

By condition (C1) the rhs is $O(1)$. Next, we consider the general case where $\hat{U}(\xi)=e^{i\xi\hat{A}_1/\sqrt{N}}\cdots e^{i\xi\hat{A}_L/\sqrt{N}}$.

$$\frac{1}{\sqrt{N}}\left\|[e^{i\xi_1\hat{A}_1/\sqrt{N}}\cdots e^{i\xi_L\hat{A}_L/\sqrt{N}},\hat{B}]\right\|\le\frac{1}{\sqrt{N}}\sum_{j=1}^{L}\left\|[e^{i\xi_j\hat{A}_j/\sqrt{N}},\hat{B}]\right\|$$
$$\le\frac{1}{N}\sum_{j=1}^{L}|\xi_j|\left\|[\hat{A}_j,\hat{B}]\right\|.$$

Again, the rhs is $O(1)$ by condition (C1). □

Lemma 2. *Assume that the system satisfies condition (C1). Let $\hat{U}(\boldsymbol{\xi})$ be a unitary operator generated by fluctuation, and $\hat{B}_1$, $\hat{B}_2$ be additive operators. Then, there exist positive constants K_1, K_2 of $O(1)$ for which the following inequality holds for any $\boldsymbol{\xi}$:*

$$\frac{1}{N}\left\|[[\hat{U},\hat{B}_1],\hat{B}_2]\right\| < K_1|\boldsymbol{\xi}| + K_2\boldsymbol{\xi}\cdot\boldsymbol{\xi}. \tag{III.17}$$

Proof. First, we prove the lemma for the case $\hat{U}(\xi) = e^{i\xi\Delta\hat{A}/\sqrt{N}}$.

$$\begin{aligned}
&\frac{1}{N}\left\|[[e^{i\xi\Delta\hat{A}/\sqrt{N}},\hat{B}_1],\hat{B}_2]\right\| \\
&= \frac{1}{N^{3/2}}\left\|[\int_0^\xi d\xi' i e^{i(\xi-\xi')\Delta\hat{A}/\sqrt{N}}[\hat{B}_1,\hat{A}]e^{i\xi'\Delta\hat{A}/\sqrt{N}},\hat{B}_2]\right\| \\
&\le \frac{1}{N^{3/2}}\left\|\int_0^\xi d\xi' i[e^{i(\xi-\xi')\Delta\hat{A}/\sqrt{N}},\hat{B}_2][\hat{B}_1,\hat{A}]e^{i\xi'\Delta\hat{A}/\sqrt{N}}\right\| + \frac{1}{N^{3/2}}\left\|\int_0^\xi d\xi' i e^{i(\xi-\xi')\Delta\hat{A}/\sqrt{N}}[[\hat{B}_1,\hat{A}],\hat{B}_2]e^{i\xi'\Delta\hat{A}/\sqrt{N}}\right\| \\
&\qquad + \frac{1}{N^{3/2}}\left\|\int_0^\xi d\xi' i e^{i(\xi-\xi')\Delta\hat{A}/\sqrt{N}}[\hat{B}_1,\hat{A}][e^{i\xi'\Delta\hat{A}/\sqrt{N}},\hat{B}_2]\right\| \\
&\le \frac{1}{N^2}\int_0^\xi d\xi'|\xi-\xi'|\left\|[\hat{A},\hat{B}_2]\right\|\left\|[\hat{B}_1,\hat{A}]\right\| + \frac{1}{N^{3/2}}\int_0^\xi d\xi'\left\|[[\hat{B}_1,\hat{A}],\hat{B}_2]\right\| + \frac{1}{N^2}\int_0^\xi d\xi'\left\|[\hat{B}_1,\hat{A}]\right\||\xi'|\left\|[\hat{A},\hat{B}_2]\right\| \\
&\le \frac{\xi^2}{2}\frac{1}{N^2}\left\|[\hat{A},\hat{B}_2]\right\|\left\|[\hat{B}_1,\hat{A}]\right\| + |\xi|\frac{1}{N^{3/2}}\left\|[[\hat{B}_1,\hat{A}],\hat{B}_2]\right\|.
\end{aligned}$$

The first term of the rhs is $O(1)$ and the second term is $o(1)$ by condition (C1). Next, we consider the general case where $\hat{U}(\boldsymbol{\xi}) = e^{i\xi_1\hat{A}_1/\sqrt{N}}\cdots e^{i\xi_L\hat{A}_L/\sqrt{N}}$.

$$\begin{aligned}
&\frac{1}{N}\left\|[[e^{i\xi_1\hat{A}_1/\sqrt{N}}\cdots e^{i\xi_L\hat{A}_L/\sqrt{N}},\hat{B}_1],\hat{B}_2]\right\| \\
&\le \frac{1}{N}\sum_{j=1}^{L}\left\|[e^{i\xi_1\hat{A}_1/\sqrt{N}}\cdots e^{i\xi_{j-1}\hat{A}_{j-1}/\sqrt{N}}[e^{i\xi_j\hat{A}_j/\sqrt{N}},\hat{B}_1]e^{i\xi_{j+1}\hat{A}_{j+1}/\sqrt{N}}\cdots e^{i\xi_L\hat{A}_L/\sqrt{N}},\hat{B}_2]\right\| \\
&\le \frac{1}{N}\sum_{j=1}^{L}\Bigg[\left\|[e^{i\xi_1\hat{A}_1/\sqrt{N}}\cdots e^{i\xi_{j-1}\hat{A}_{j-1}/\sqrt{N}},\hat{B}_2]\right\|\left\|[e^{i\xi_j\hat{A}_j/\sqrt{N}},\hat{B}_1]\right\| \\
&\qquad + \left\|[[e^{i\xi_j\hat{A}_j/\sqrt{N}},\hat{B}_1],\hat{B}_2]\right\| \\
&\qquad + \left\|[e^{i\xi_j\hat{A}_j/\sqrt{N}},\hat{B}_1]\right\|\left\|[e^{i\xi_{j+1}\hat{A}_{j+1}/\sqrt{N}}\cdots e^{i\xi_L\hat{A}_L/\sqrt{N}},\hat{B}_2]\right\|\Bigg].
\end{aligned}$$

By Lemma 1 and the previous case, Lemma 2 is proved. □

Lemma 3 (Bound for the first derivative of the 'characteristic function')**.** *Assume that the system satisfies conditions (C1) and (C2). Let $\hat{U}_1(\boldsymbol{\xi})$, $\hat{U}_2(\boldsymbol{\xi}')$ be unitary operators generated by fluctuation, and $\hat{B}$ an additive operator. Then, there exist positive constants K_0 and K_1 of $O(1)$ for which the following inequality holds for any $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$:*

$$\frac{1}{\sqrt{N}}|\langle\hat{U}_1(\boldsymbol{\xi})\Delta\hat{B}\hat{U}_2(\boldsymbol{\xi}')\rangle_{\rm eq}| < K_0 + K_1|\boldsymbol{\xi}|. \tag{III.18}$$

Proof.

$$\begin{aligned}
\frac{1}{\sqrt{N}}\left|\langle\hat{U}_1(\boldsymbol{\xi})\Delta\hat{B}\hat{U}_2(\boldsymbol{\xi}')\rangle_{\rm eq}\right| &\le \frac{1}{\sqrt{N}}|\langle\Delta\hat{B}\hat{U}_1(\boldsymbol{\xi})\hat{U}_2(\boldsymbol{\xi}')\rangle_{\rm eq}| + \frac{1}{\sqrt{N}}|\langle[\hat{U}_1(\boldsymbol{\xi}),\Delta\hat{B}]\hat{U}_2(\boldsymbol{\xi}')\rangle_{\rm eq}| \\
&\le \frac{1}{\sqrt{N}}\langle\Delta\hat{B}^2\rangle_{\rm eq}^{1/2} + \frac{1}{\sqrt{N}}\left\|[\hat{U}_1(\boldsymbol{\xi}),\Delta\hat{B}]\right\|.
\end{aligned}$$

The last line follows from the Cauchy-Schwarz inequality. The first term is bounded by a constant of $O(1)$ by condition (C2). Also, the second term is bounded by $|\boldsymbol{\xi}|$ times a constant of $O(1)$ by Lemma 1. □

Lemma 4 (Bound for the second derivative of the 'characteristic function'). *Assume that the system satisfies conditions (C1) and (C2). Let $\hat{U}_1(\boldsymbol{\xi}_1)$, $\hat{U}_2(\boldsymbol{\xi}_2)$, $\hat{U}_3(\boldsymbol{\xi}_3)$ be unitary operators generated by fluctuation, and $\hat{B}_1$, $\hat{B}_2$ be additive operators. Then, there exist positive constants K_0, K_1 and K_2 of $O(1)$ for which the following inequality holds for any $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$:*

$$\frac{1}{N}|\langle\hat{U}_1(\boldsymbol{\xi}_1)\Delta\hat{B}_1\hat{U}_2(\boldsymbol{\xi}_2)\Delta\hat{B}_2\hat{U}_3(\boldsymbol{\xi}_3)\rangle_{\rm eq}| \leq K_0 + K_1(|\boldsymbol{\xi}_1|+|\boldsymbol{\xi}_3|) + K_2|\boldsymbol{\xi}_1||\boldsymbol{\xi}_3|. \tag{III.19}$$

Proof.

$$\begin{aligned}
&\frac{1}{N}|\langle\hat{U}_1(\boldsymbol{\xi}_1)\Delta\hat{B}_1\hat{U}_2(\boldsymbol{\xi}_2)\Delta\hat{B}_2\hat{U}_3(\boldsymbol{\xi}_3)\rangle_{\rm eq}|\\
&\leq \frac{1}{N}|\langle\Delta\hat{B}_1\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)\Delta\hat{B}_2\rangle_{\rm eq}| + \frac{1}{N}|\langle[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)\Delta\hat{B}_2\rangle_{\rm eq}|\\
&\qquad + \frac{1}{N}|\langle\Delta\hat{B}_1\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2)[\Delta\hat{B}_2,\hat{U}_3(\boldsymbol{\xi}_3)]\rangle_{\rm eq}| + \frac{1}{N}|\langle[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\hat{U}_2(\boldsymbol{\xi}_2)[\Delta\hat{B}_2,\hat{U}_3(\boldsymbol{\xi}_3)]\rangle_{\rm eq}|\\
&\leq \frac{1}{N}\langle\Delta\hat{B}_1^2\rangle_{\rm eq}^{1/2}\langle\Delta\hat{B}_2^2\rangle_{\rm eq}^{1/2} + \frac{1}{N}\left\|[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\right\|\langle\Delta\hat{B}_2^2\rangle_{\rm eq}^{1/2}\\
&\qquad + \frac{1}{N}\langle\Delta\hat{B}_1^2\rangle_{\rm eq}^{1/2}\left\|[\Delta\hat{B}_2,\hat{U}_3(\boldsymbol{\xi}_3)]\right\| + \frac{1}{N}\left\|[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\right\|\left\|[\Delta\hat{B}_2,\hat{U}_3(\boldsymbol{\xi}_3)]\right\|.
\end{aligned}$$

The last line follows from the Cauchy-Schwarz inequality. From condition (C2) and Lemma 1, Eq. (III.19) holds. □

Lemma 5 (Bound for the third derivative of the 'characteristic function'). *Assume that the system satisfies conditions (C1), (C2) and (C3). Let $\hat{U}_1(\boldsymbol{\xi}_1)$, $\hat{U}_2(\boldsymbol{\xi}_2)$, $\hat{U}_3(\boldsymbol{\xi}_3)$, $\hat{U}_4(\boldsymbol{\xi}_4)$ be unitary operators generated by fluctuation, and $\hat{B}_1$, $\hat{B}_2$, $\hat{B}_3$ be additive operators. Then, there exist positive constants K_0, K_1, K_2 and K_3 of $O(1)$ for which the following inequality holds for any $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$ and $\boldsymbol{\xi}_4$:*

$$\begin{aligned}
&\frac{1}{N^{3/2}}\left|\langle\hat{U}_1(\boldsymbol{\xi}_1)\Delta\hat{B}_1\hat{U}_2(\boldsymbol{\xi}_2)\Delta\hat{B}_2\hat{U}_3(\boldsymbol{\xi}_3)\Delta\hat{B}_3\hat{U}_4(\boldsymbol{\xi}_4)\rangle_{\rm eq}\right|\\
&\quad\leq K_0 + K_1(|\boldsymbol{\xi}_1|+|\boldsymbol{\xi}_2|+|\boldsymbol{\xi}_4|) + K_2(|\boldsymbol{\xi}_1||\boldsymbol{\xi}_3|+|\boldsymbol{\xi}_1||\boldsymbol{\xi}_4|+|\boldsymbol{\xi}_2||\boldsymbol{\xi}_4|) + K_3(|\boldsymbol{\xi}_1|+|\boldsymbol{\xi}_2|)|\boldsymbol{\xi}_1||\boldsymbol{\xi}_4|.
\end{aligned} \tag{III.20}$$

Proof.

$$\begin{aligned}
&\frac{1}{N^{3/2}}|\langle\hat{U}_1(\boldsymbol{\xi}_1)\Delta\hat{B}_1\hat{U}_2(\boldsymbol{\xi}_2)\Delta\hat{B}_2\hat{U}_3(\boldsymbol{\xi}_3)\Delta\hat{B}_3\hat{U}_4(\boldsymbol{\xi}_4)\rangle_{\rm eq}|\\
&\leq \frac{1}{N^{3/2}}|\langle\Delta\hat{B}_1\Delta\hat{B}_2\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)\hat{U}_4(\boldsymbol{\xi}_4)\Delta\hat{B}_3\rangle_{\rm eq}| + \frac{1}{N^{3/2}}|\langle\Delta\hat{B}_1[\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2),\Delta\hat{B}_2]\hat{U}_3(\boldsymbol{\xi}_3)\hat{U}_4(\boldsymbol{\xi}_4)\Delta\hat{B}_3\rangle_{\rm eq}|\\
&\quad + \frac{1}{N^{3/2}}|\langle[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)\hat{U}_4(\boldsymbol{\xi}_4)\Delta\hat{B}_2\Delta\hat{B}_3\rangle_{\rm eq}| + \frac{1}{N^{3/2}}|\langle[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\hat{U}_2(\boldsymbol{\xi}_2)[\Delta\hat{B}_2,\hat{U}_3(\boldsymbol{\xi}_3)\hat{U}_4(\boldsymbol{\xi}_4)]\Delta\hat{B}_3\rangle_{\rm eq}|\\
&\quad + \frac{1}{N^{3/2}}|\langle\Delta\hat{B}_1\Delta\hat{B}_2\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\rangle_{\rm eq}| + \frac{1}{N^{3/2}}|\langle\Delta\hat{B}_1[\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2),\Delta\hat{B}_2]\hat{U}_3(\boldsymbol{\xi}_3)[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\rangle_{\rm eq}|\\
&\quad + \frac{1}{N^{3/2}}|\langle\Delta\hat{B}_2[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\rangle_{\rm eq}| + \frac{1}{N^{3/2}}|\langle[[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1],\Delta\hat{B}_2]\hat{U}_2(\boldsymbol{\xi}_2)\hat{U}_3(\boldsymbol{\xi}_3)[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\rangle_{\rm eq}|\\
&\quad + \frac{1}{N^{3/2}}|\langle[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1][\hat{U}_2(\boldsymbol{\xi}_2),\Delta\hat{B}_2]\hat{U}_3(\boldsymbol{\xi}_3)[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\rangle_{\rm eq}|\\
&\leq \frac{1}{N^{3/2}}\langle\Delta\hat{B}_1\Delta\hat{B}_2^2\Delta\hat{B}_1\rangle_{\rm eq}^{1/2}\langle\Delta\hat{B}_3^2\rangle_{\rm eq}^{1/2} + \frac{1}{N^{3/2}}\left\|[\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2),\Delta\hat{B}_2]\right\|\langle\Delta\hat{B}_1^2\rangle_{\rm eq}^{1/2}\langle\Delta\hat{B}_3^2\rangle_{\rm eq}^{1/2}\\
&\quad + \frac{1}{N^{3/2}}\left\|[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\right\|\langle\Delta\hat{B}_3\Delta\hat{B}_2^2\Delta\hat{B}_3\rangle_{\rm eq}^{1/2} + \frac{1}{N^{3/2}}\left\|[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\right\|\left\|[\Delta\hat{B}_2,\hat{U}_3(\boldsymbol{\xi}_3)\hat{U}_4(\boldsymbol{\xi}_4)]\right\|\langle\Delta\hat{B}_3^2\rangle_{\rm eq}^{1/2}\\
&\quad + \frac{1}{N^{3/2}}\langle\Delta\hat{B}_1\Delta\hat{B}_2^2\Delta\hat{B}_1\rangle_{\rm eq}^{1/2}\left\|[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\right\| + \frac{1}{N^{3/2}}\left\|[\hat{U}_1(\boldsymbol{\xi}_1)\hat{U}_2(\boldsymbol{\xi}_2),\Delta\hat{B}_2]\right\|\left\|[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\right\|\langle\Delta\hat{B}_1^2\rangle_{\rm eq}^{1/2}\\
&\quad + \frac{1}{N^{3/2}}\left\|[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\right\|\left\|[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\right\|\langle\Delta\hat{B}_2^2\rangle_{\rm eq}^{1/2} + \frac{1}{N^{3/2}}\left\|[[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1],\Delta\hat{B}_2]\right\|\left\|[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\right\|\\
&\quad + \frac{1}{N^{3/2}}\left\|[\hat{U}_1(\boldsymbol{\xi}_1),\Delta\hat{B}_1]\right\|\left\|[\hat{U}_2(\boldsymbol{\xi}_2),\Delta\hat{B}_2]\right\|\left\|[\Delta\hat{B}_3,\hat{U}_4(\boldsymbol{\xi}_4)]\right\|.
\end{aligned}$$

The last line follows from the Cauchy-Schwarz inequality and the definition of the operator norm. By conditions (C2) and (C3) and Lemmas 1 and 2, Eq. (III.20) is proved. □

Lemma 6. *Let I be a closed and bounded interval, and $\{g_N(x)\}_N$ be a sequence of twice differentiable functions on I which converges (pointwise) to $g(x)$. Assume that the sequence of second derivatives $\{g''_N(x)\}_N$ are uniformly bounded on I. Then, $\{g'_N(x)\}_N$ converges uniformly to $g'(x)$ (hence we can interchange the differential operation and the limit operation).*

Proof. We use a corollary of the Arzelà-Ascoli theorem : *If a sequence of functions $\{h_N(x)\}_N$ on a closed and bounded interval is uniformly bounded and equicontinuous, and if any subsequences (of the sequence) which converge uniformly have the same limit function $h(x)$ independent of the choice of the subsequence, then the original sequence $\{h_N(x)\}_N$ also converges uniformly to $h(x)$.* Now, by the condition, the sequence of the first derivatives $\{g'_N(x)\}_N$ is uniformly bounded and equicontinuous on I. Additionally, if a subsequence $\{g'_{N_k}(x)\}_k$ $(N_1 < N_2 < \cdots)$ converges uniformly, its limit is $g'(x)$, independent of $\{N_k\}_k$. Therefore, $\{g'_N(x)\}_N$ converges uniformly to $g'(x)$. □

Now, we can prove the theorems using these lemmas.

Proof of Theorems 2 and 3. We fix ξ and a closed bounded interval I including ξ. By Lemma 4, the second derivative of $\langle e^{i\xi_1\Delta\hat{A}_1/\sqrt{N}}\dots e^{i\xi_1\Delta\hat{A}_1/\sqrt{N}}\rangle_{\rm eq}$ with respect to ξ_i is uniformly bounded on the interval. We can then apply Lemma 6, hence Theorem 2 holds. Similarly, by Lemma 5, the third derivative of $\langle e^{i\xi_1\Delta\hat{A}_1/\sqrt{N}}\dots e^{i\xi_1\Delta\hat{A}_1/\sqrt{N}}\rangle_{\rm eq}$ with respect to ξ_i is uniformly bounded on the interval, hence Theorem 3 holds. □

C. Some more theorems

In this subsection, we present some more theorems which will also be used in the next section for deriving our results. The theorems will be proved under certain conditions on $f(x)$, which will be sufficient conditions for our results. We also prove that Schwartz functions satisfy all the conditions. As in the main paper, we write

$$\hat{a} = \hat{A}/\sqrt{N}, \tag{III.21}$$

$$\Delta\hat{a} = \hat{a} - \langle\hat{a}\rangle_{\rm eq}. \tag{III.22}$$

Note that a function of a self-adjoint operator, such as $f(\hat{a} - a_\bullet)$, is defined in terms of the spectrum decomposition of the operator.

Theorem 4 ('characteristic function' of squeezed equilibrium state)**.** *If the conditions of Sec. III A are satisfied for $\hat{A}_0,\dots,\hat{A}_M,\hat{B}$, and if $f_0,\ \dots,\ f_M$ are bounded piecewise-continuous real functions, then*

$$\begin{aligned}
&\lim_{N\to\infty}\langle f_0(\hat{a}^0 - a^0_\bullet)\cdots f_M(\hat{a}^M - a^M_\bullet)e^{iu\Delta\hat{b}}f_M(\hat{a}^M - a^M_\bullet)\cdots f_0(\hat{a}^M - a^M_\bullet)\rangle_{\rm eq}\\
&= \int\frac{d\xi_0 d\xi'_0}{2\pi}\cdots\frac{d\xi_M d\xi'_M}{2\pi}f_0[\xi]f_0[\xi']\cdots f_M[\xi]f_M[\xi']\\
&\qquad\times\lim_{N\to\infty}\langle e^{i\xi_0(\hat{a}^0-a^0_\bullet)}\dots e^{i\xi_M(\hat{a}^M-a^M_\bullet)}e^{iu\Delta\hat{b}}e^{i\xi'_M(\hat{a}^M-a^M_\bullet)}\dots e^{i\xi'_0(\hat{a}^0-a^0_\bullet)}\rangle_{\rm eq}
\end{aligned}\tag{III.23}$$

$$\begin{aligned}
&= \int\frac{d\xi_0 d\xi'_0}{2\pi}\cdots\frac{d\xi_M d\xi'_M}{2\pi}f_0[\xi]f_0[\xi']\cdots f_M[\xi]f_M[\xi']\\
&\qquad\times\exp\Bigg[-i\sum_{k=0}^{M}(\xi_k+\xi'_k)\Delta a^k_\bullet - \frac{1}{2}\sum_{k,l}^{M}(\xi_k+\xi'_k)(\xi_l+\xi'_l)\lim_{N\to\infty}\langle\frac{1}{2}\{\Delta\hat{a}^k(t_k),\Delta\hat{a}^l(t_l)\}\rangle_{\rm eq}\\
&\qquad\qquad - \frac{1}{2}u^2\lim_{N\to\infty}\langle\Delta\hat{b}^2\rangle_{\rm eq} - \frac{u}{2}\sum_{k=0}^{M}\xi_k\lim_{N\to\infty}\langle\frac{1}{2}\{\Delta\hat{b},\Delta\hat{a}^k(t_k)\}\rangle_{\rm eq}\\
&\qquad\qquad - \sum_{k<l}^{M}(\xi_k-\xi'_k)(\xi_l+\xi'_l)\lim_{N\to\infty}\langle\frac{1}{2}[\hat{a}^k(t_k),\hat{a}^l(t_l)]\rangle_{\rm eq} - u\sum_{k=0}^{M}(\xi_k-\xi'_k)\lim_{N\to\infty}\langle\frac{1}{2}[\hat{a}^k(t_k),\hat{b}]\rangle_{\rm eq}\Bigg].
\end{aligned}\tag{III.24}$$

Here, $f_k[\xi]$ is the Fourier transform of $f_k(x)$:

$$f_k[\xi] = \int\frac{d\xi}{\sqrt{2\pi}}e^{-i\xi x}f_k(x). \tag{III.25}$$

Proof. By Theorem 2 in [5], the limit of expectation value in the lhs of Eq. (III.23) is determined by the QCLT, and the limit can be expressed by the Fourier transform as (III.24). □

Theorem 5 (expectation value in squeezed equilibrium state)**.** *Assume that the conditions of Sec. III A are satisfied for* $\hat{A}_0, \ldots, \hat{A}_M, \hat{B}$, *and that* $f_0, \ldots, f_M$ *are bounded piecewise-continuous real functions. If*

$$\langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^2 f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \leq \textit{an upper bound independent of } N, \tag{III.26}$$

then

$$\begin{aligned} &\lim_{N\to\infty} \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \\ &= -i \frac{\partial}{\partial u} \lim_{N\to\infty} \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) e^{iu\Delta\hat{b}} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \big|_{u=0} \end{aligned} \tag{III.27}$$

Proof. By the Cauchy-Schwarz inequality and (III.26),

$$\begin{aligned} &\left| \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^2 e^{iu\hat{b}} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \right| \\ &\leq \left| \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^2 f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \right| \\ &\leq \text{an upper bound independent of } N. \end{aligned}$$

This implies that the second derivative of the 'characteristic function'

$$\frac{\partial^2}{\partial u^2} \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) e^{iu\hat{b}} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \tag{III.28}$$

is uniformly bounded on any bounded interval of u. Hence, by Theorem 4 and Lemma 6, Eq. (III.27) is proved. □

Theorem 6 (variance in the squeezed equilibrium state)**.** *Assume that the conditions of Sec. III A are satisfied for* $\hat{A}_0, \ldots, \hat{A}_M, \hat{B}$, *and that* $f_0, \ldots, f_M$ *are bounded piecewise-continuous real functions. If*

$$\langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^4 f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \leq \textit{an upper bound independent of } N, \tag{III.29}$$

then

$$\begin{aligned} &\lim_{N\to\infty} \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^2 f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \\ &= -\frac{\partial^2}{\partial u^2} \lim_{N\to\infty} \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) e^{iu\Delta\hat{b}} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \big|_{u=0} \end{aligned} \tag{III.30}$$

Proof. By the Cauchy-Schwarz inequality and (III.29),

$$\begin{aligned} &\left| \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^3 e^{iu\hat{b}} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \right| \\ &\leq \left| \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^2 f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \right|^{1/2} \\ &\qquad \times \left| \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) \Delta\hat{b}^4 f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \right|^{1/2} \\ &\leq \text{an upper bound independent of } N. \end{aligned}$$

This implies that the third derivative of the 'characteristic function'

$$\frac{\partial^3}{\partial u^3} \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_M(\hat{a}^M - a^M_\bullet) e^{iu\hat{b}} f_M(\hat{a}^M - a^M_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet) \rangle_{\rm eq} \tag{III.31}$$

is uniformly bounded on any bounded interval of u. Hence, by Theorem 4 and Lemma 6, Eq. (III.27) is proved. □

As an example of a class of functions that satisfy the conditions of the above theorems, let us consider a Schwartz function which is defined as follows.

Definition 1 (Schwartz function [6–9])**.** A *Schwartz function* is a infinitely differentiable function such that for any $n, m = 0, 1, \ldots,$

$$\sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)| < \infty. \tag{III.32}$$

For example, $x^k e^{-x^2}$ is a Schwartz function for any $k > 0$.

A Schwartz function has good properties, such as (i) a Schwartz function is absolutely integrable on $\mathbb{R}$, (ii) the Fourier transform of a Schwartz function is also a Schwartz function. As a result, Schwartz functions satisfy all the conditions of the above theorems.

Theorem 7. *Assume that the conditions of Sec. III A are satisfied for $\hat{A}_0, \ldots, \hat{A}_M, \hat{B}$. For any Schwartz function f, all the conditions of Theorems 5 and 6 are satisfied.*

Proof. By Lemma 4, there exist positive constants K_0, K_1 and K_2 such that

$$\begin{aligned}
&\left|\langle f_0(\hat{a}^0 - a^0_\bullet)\cdots f_M(\hat{a}^M - a^M_\bullet)\Delta\hat{b}^2 f_M(\hat{a}^M - a^M_\bullet)\cdots f_0(\hat{a}^M - a^M_\bullet)\rangle_{\mathrm{eq}}\right| \\
&\le \int \frac{d\xi_0 d\xi'_0}{2\pi}\cdots\frac{d\xi_M d\xi'_M}{2\pi}\,|f_0[\xi]f_0[\xi']\cdots f_M[\xi]f_M[\xi']|\left|\langle e^{i\xi_0(\hat{a}^0-a^0_\bullet)}\cdots e^{i\xi_M(\hat{a}^M-a^M_\bullet)}\Delta\hat{b}^2 e^{i\xi'_M(\hat{a}^M-a^M_\bullet)}\cdots e^{i\xi'_0(\hat{a}^0-a^0_\bullet)}\rangle_{\mathrm{eq}}\right| \\
&\le \int \frac{d\xi_0 d\xi'_0}{2\pi}\cdots\frac{d\xi_M d\xi'_M}{2\pi}\,|f_0[\xi]f_0[\xi']\cdots f_M[\xi]f_M[\xi']|\left[K_0 + K_1\sum_j(|\xi_j| + |\xi'_j|) + K_2\sum_{j,l}|\xi_j||\xi_l|\right].
\end{aligned}$$

Since $i\xi_j f_j[\xi_j]$ is the Fourier transform of $f'_j(x)$, it is a Schwartz function. Hence, the integrand above is absolutely integrable, and Eq. (III.26) is satisfied. Eq. (III.29) is also proved similarly. □

Next, we prove theorems for the average value.

Theorem 8. *If $f_j \in L^2(\mathbb{R})$ and $\xi|f_j[\xi]|^2$ are absolutely integrable then*

$$\begin{aligned}
&\lim_{N\to\infty}\int da^0_\bullet\cdots da^M_\bullet\langle f_0(\hat{a}^0 - a^0_\bullet)\cdots f_M(\hat{a}^M - a^M_\bullet)\Delta\hat{b} f_M(\hat{a}^M - a^M_\bullet)\cdots f_0(\hat{a}^M - a^M_\bullet)\rangle_{\mathrm{eq}} \\
&= \int d\xi_0\cdots d\xi_M |f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2 \lim_{N\to\infty}\langle e^{i\xi_0\Delta\hat{a}^0}\cdots e^{i\xi_M\Delta\hat{a}^M}\Delta\hat{b}e^{-i\xi_M\Delta\hat{a}^M}\cdots e^{-i\xi_0\Delta\hat{a}^0}\rangle_{\mathrm{eq}}.
\end{aligned} \tag{III.33}$$

Note that the conditions of this theorem are satisfied if, for example, f_j are continuous and piecewise differentiable and f'_j (on each of their subdomains) are square integrable.

Proof.

$$\begin{aligned}
&\int da^0_\bullet\cdots da^M_\bullet\langle f_0(\hat{a}^0 - a^0_\bullet)\cdots f_M(\hat{a}^M - a^M_\bullet)\Delta\hat{b} f_M(\hat{a}^M - a^M_\bullet)\cdots f_0(\hat{a}^M - a^M_\bullet)\rangle_{\mathrm{eq}} \\
&= \int da^0_\bullet\cdots da^M_\bullet\int\frac{d\xi_0 d\xi'_0}{2\pi}\cdots\frac{d\xi_M d\xi'_M}{2\pi} f_0[\xi_0]f^*_0[\xi'_0]\cdots f_M[\xi_M]f^*_M[\xi'_M] \\
&\qquad\times\langle e^{i\xi_0(\Delta\hat{a}^0-\Delta a^0_\bullet)}\cdots e^{i\xi_M(\Delta\hat{a}^M-\Delta a^M_\bullet)}\Delta\hat{b}e^{-i\xi'_M(\Delta\hat{a}^M-\Delta a^M_\bullet)}\cdots e^{-i\xi'_0(\Delta\hat{a}^0-\Delta a^0_\bullet)}\rangle_{\mathrm{eq}} \\
&= \int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\langle e^{i\xi_0\Delta\hat{a}^0}\cdots e^{i\xi_M\Delta\hat{a}^M}\Delta\hat{b}e^{-i\xi_M\Delta\hat{a}^M}\cdots e^{-i\xi_0\Delta\hat{a}^0}\rangle_{\mathrm{eq}}.
\end{aligned}$$

By Lemma 3, there are $O(1)$-positive constants K_0 and K_1, and the integrand of the last integral is bounded by

$$\begin{aligned}
&|f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\left|\langle e^{i\xi_0\Delta\hat{a}^0}\cdots e^{i\xi_M\Delta\hat{a}^M}\Delta\hat{b}e^{-i\xi_M\Delta\hat{a}^M}\cdots e^{-i\xi_0\Delta\hat{a}^0}\rangle_{\mathrm{eq}}\right| \\
&\le |f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\left(K_0 + K_1\sum_{j=0}^M|\xi_j|\right).
\end{aligned}$$

By assumption and the fact that the Fourier transform of any $L^2(\mathbb{R})$ function is also an $L^2(\mathbb{R})$ function, the above upper bound is absolutely integrable:

$$\int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\left(K_0 + K_1\sum_{j=0}^M|\xi_j|\right) < \infty.$$

Hence, by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{N\to\infty}\int da_\bullet^0\cdots da_\bullet^M \langle f_0(\hat a^0 - a_\bullet^0)\cdots f_M(\hat a^M - a_\bullet^M)\Delta\hat b f_M(\hat a^M - a_\bullet^M)\cdots f_0(\hat a^M - a_\bullet^M)\rangle_{\rm eq}\\
&= \lim_{N\to\infty}\int d\xi_0\cdots d\xi_M |f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_M\Delta\hat a^M}\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_0\Delta\hat a^0}\rangle_{\rm eq}\\
&= \int d\xi_0\cdots d\xi_M |f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2 \lim_{N\to\infty}\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_M\Delta\hat a^M}\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_0\Delta\hat a^0}\rangle_{\rm eq}.
\end{aligned}$$

□

Theorem 9. *If $f_j \in L^2(\mathbb{R})$ and $f_j[\xi]$ are continuous and piecewise differentiable, and if $\xi f_j'[\xi]f_j[\xi]$ and $\xi^2|f_j[\xi]|^2$ are absolutely integrable, then*

$$\begin{aligned}
&\lim_{N\to\infty}\int da_\bullet^0\cdots da_\bullet^M\ \Delta a_\bullet^l\langle f_0(\hat a^0 - a_\bullet^0)\cdots f_M(\hat a^M - a_\bullet^M)\Delta\hat b f_M(\hat a^M - a_\bullet^M)\cdots f_0(\hat a^M - a_\bullet^M)\rangle_{\rm eq}\\
&= \int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_{l-1}[\xi_{l-1}]|^2\,{\rm Im}(f_l'[\xi_l]f_l^*[\xi_l])|f_{l+1}[\xi_{l+1}]|^2\cdots|f_M[\xi_M]|^2\\
&\qquad\times\lim_{N\to\infty}\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_M\Delta\hat a^M}\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_0\Delta\hat a^0}\rangle_{\rm eq}\\
&\quad - i\int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\\
&\qquad\times\lim_{N\to\infty}\frac{\partial}{\partial s}\Big\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_{l-1}\Delta\hat a^{l-1}}e^{i(\xi_l+s/2)\Delta\hat a^l}e^{i\xi_{l+1}\Delta\hat a^{l+1}}\cdots e^{i\xi_M\Delta\hat a^M}\\
&\qquad\qquad\times\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_{l+1}\Delta\hat a^{l+1}}e^{-i(\xi_l+s/2)\Delta\hat a^l}e^{-i\xi_{l-1}\Delta\hat a^{l-1}}\cdots e^{-i\xi_0\Delta\hat a^0}\Big\rangle_{\rm eq}\Big|_{s=0}.
\end{aligned}$$

Note that the conditions of this theorem are satisfied if, for example, f_j are continuously differentiable and piecewise twice differentiable, and f_j, xf_j, f_j', f_j'' (on each of their subdomains) are square integrable.

Proof.

$$\begin{aligned}
&\int da_\bullet^0\cdots da_\bullet^M\ \Delta a_\bullet^l\langle f_0(\hat a^0 - a_\bullet^0)\cdots f_M(\hat a^M - a_\bullet^M)\Delta\hat b f_M(\hat a^M - a_\bullet^M)\cdots f_0(\hat a^M - a_\bullet^M)\rangle_{\rm eq}\\
&= \int da_\bullet^0\cdots da_\bullet^M\ \Delta a_\bullet^l\int\frac{d\xi_0 d\xi_0'}{2\pi}\cdots\frac{d\xi_M d\xi_M'}{2\pi}f_0[\xi_0]f_0^*[\xi_0']\cdots f_M[\xi_M]f_M^*[\xi_M']\\
&\qquad\times\langle e^{i\xi_0(\Delta\hat a^0-\Delta a_\bullet^0)}\cdots e^{i\xi_M(\Delta\hat a^M-\Delta a_\bullet^M)}\Delta\hat b e^{-i\xi_M'(\Delta\hat a^M-\Delta a_\bullet^M)}\cdots e^{-i\xi_0'(\Delta\hat a^0-\Delta a_\bullet^0)}\rangle_{\rm eq}\\
&= i\int da_\bullet^0\cdots da_\bullet^M\int\frac{d\xi_0 d\xi_0'}{2\pi}\cdots\frac{d\xi_M d\xi_M'}{2\pi}\frac{\partial}{\partial(\xi_l-\xi_l')}\Big\{f_0[\xi_0]f_0^*[\xi_0']\cdots f_M[\xi_M]f_M^*[\xi_M']\\
&\qquad\times\langle e^{i\xi_0(\Delta\hat a^0-\Delta a_\bullet^0)}\cdots e^{i\xi_M(\Delta\hat a^M-\Delta a_\bullet^M)}\Delta\hat b e^{-i\xi_M'(\Delta\hat a^M-\Delta a_\bullet^M)}\cdots e^{-i\xi_0'(\Delta\hat a^0-\Delta a_\bullet^0)}\rangle_{\rm eq}\Big\}\\
&= -i\frac{\partial}{\partial s}\int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_{l-1}[\xi_{l-1}]|^2 f_l[\xi_l+\frac{s}{2}]f_l^*[\xi_l-\frac{s}{2}]|f_{l+1}[\xi_{l+1}]|^2\cdots|f_M[\xi_M]|^2\\
&\qquad\times\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_{l-1}\Delta\hat a^{l-1}}e^{i(\xi_l+s/2)\Delta\hat a^l}e^{i\xi_{l+1}\Delta\hat a^{l+1}}\cdots e^{i\xi_M\Delta\hat a^M}\\
&\qquad\qquad\times\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_{l+1}\Delta\hat a^{l+1}}e^{-i(\xi_l-s/2)\Delta\hat a^l}e^{-i\xi_{l-1}\Delta\hat a^{l-1}}\cdots e^{-i\xi_0\Delta\hat a^0}\rangle_{\rm eq}\big|_{s=0}\\
&= \int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_{l-1}[\xi_{l-1}]|^2\,{\rm Im}(f_l'[\xi_l]f_l^*[\xi_l])|f_{l+1}[\xi_{l+1}]|^2\cdots|f_M[\xi_M]|^2\\
&\qquad\times\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_M\Delta\hat a^M}\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_0\Delta\hat a^0}\rangle_{\rm eq}\\
&\quad - i\int d\xi_0\cdots d\xi_M|f_0[\xi_0]|^2\cdots|f_M[\xi_M]|^2\\
&\qquad\times\frac{\partial}{\partial s}\Big\langle e^{i\xi_0\Delta\hat a^0}\cdots e^{i\xi_{l-1}\Delta\hat a^{l-1}}e^{i(\xi_l+s/2)\Delta\hat a^l}e^{i\xi_{l+1}\Delta\hat a^{l+1}}\cdots e^{i\xi_M\Delta\hat a^M}\\
&\qquad\qquad\times\Delta\hat b e^{-i\xi_M\Delta\hat a^M}\cdots e^{-i\xi_{l+1}\Delta\hat a^{l+1}}e^{-i(\xi_l+s/2)\Delta\hat a^l}e^{-i\xi_{l-1}\Delta\hat a^{l-1}}\cdots e^{-i\xi_0\Delta\hat a^0}\Big\rangle_{\rm eq}\Big|_{s=0}.
\end{aligned}$$

By Lemmas 3 and 4, there are $O(1)$-positive constants K_0, K_1 and K_2, and the integrand above is bounded by

$$|f_0[\xi_0]|^2 \cdots |f_{l-1}[\xi_{l-1}]|^2 \, |\mathrm{Im}(f_l'[\xi_l] f_l^*[\xi_l])| \, |f_{l+1}[\xi_{l+1}]|^2 \cdots |f_M[\xi_M]|^2 \left(K_0 + K_1 \sum_{j=0}^{M} |\xi_j| \right)$$
$$+ |f_0[\xi_0]|^2 \cdots |f_M[\xi_M]|^2 \left(K_0 + K_1 \sum_{j=0}^{M} |\xi_j| + K_2 \left(\sum_{j=0}^{M} |\xi_j| \right)^2 \right).$$

By assumption, the above upper bound is absolutely integrable:

$$\int d\xi_0 \cdots d\xi_M |f_0[\xi_0]|^2 \cdots |f_{l-1}[\xi_{l-1}]|^2 \, |\mathrm{Im}(f_l'[\xi_l] f_l^*[\xi_l])| \, |f_{l+1}[\xi_{l+1}]|^2 \cdots |f_M[\xi_M]|^2 \left(K_0 + K_1 \sum_{j=0}^{M} |\xi_j| \right)$$
$$+ \int d\xi_0 \cdots d\xi_M |f_0[\xi_0]|^2 \cdots |f_M[\xi_M]|^2 \left(K_0 + K_1 \sum_{j=0}^{M} |\xi_j| + K_2 \left(\sum_{j=0}^{M} |\xi_j| \right)^2 \right) < \infty.$$

Hence, by the dominated convergence theorem, the theorem is proved. □

IV. DERIVATIONS OF THE MAIN RESULTS

We now derive the main results of the paper using the above theorems. We assume that the conditions of Sec. III A are satisfied for all additive observables of interest. For the conditions on f, we describe a *sufficient* condition on f for *each* result. Hence, a sufficient condition on f for *all* the results is the logical conjunction of these conditions. We think the conditions are general enough from a physical viewpoint, although it might be possible to prove the results under weaker conditions on f.

Note that Schwartz functions satisfy all the conditions on f, as is easily seen from their general properties and Thoerem 7.

A. Derivation of Eq. (3)

A sufficient condition on f for Eq. (3) is that f is a bounded piecewise-continuous real function.

By the QCLT, the probability distribution of fluctuation of any additive quantity which satisfies the conditions in Sec. III A converges weakly to the normal distribution in the TDL. Therefore, for any bounded piecewise-continuous real function f,

$$\begin{aligned} p(a_\bullet) &\equiv \lim_{N\to\infty} \langle \{ f(\hat{a} - a_\bullet) \}^2 \rangle_{\mathrm{eq}} \\ &= \int \frac{|f(a - a_\bullet)|^2}{(2\pi \delta a_{\mathrm{eq}}^2)^{1/2}} \exp\left[-\frac{(a - \langle \hat{a} \rangle_{\mathrm{eq}})^2}{2\delta a_{\mathrm{eq}}^2} \right] da \\ &= \int \frac{|f(x)|^2}{(2\pi \delta a_{\mathrm{eq}}^2)^{1/2}} \exp\left[-\frac{(x + \Delta a_\bullet)^2}{2\delta a_{\mathrm{eq}}^2} \right] dx. \end{aligned} \tag{IV.1}$$

For example, in the case where f is gaussian $f(x) = (2\pi w^2)^{-1/4} \exp(-x^2/4w^2)$,

$$p(a_\bullet) = \frac{1}{\sqrt{2\pi(w^2 + \delta a_{\mathrm{eq}}^2)}} \exp\left[-\frac{1}{2} \frac{\Delta a_\bullet^2}{w^2 + \delta a_{\mathrm{eq}}^2} \right]. \tag{IV.2}$$

B. Derivation of Eq. (4) and Eq. (13)

A sufficient condition on f for Eqs. (4) and (13) is that f is a bounded piecewise-continuous real function and that conditions (III.26) and (III.29) are satisfied.

To derive Eq. (4) and Eq. (13), we calculate the 'characteristic function' of $\Delta\hat{b}(t)$ for the state $\hat{\rho}(a_\bullet)$:

$$\langle \exp(iu\Delta\hat{b}(t))\rangle_{a_\bullet} = \frac{1}{p(a_\bullet)}\langle f(\hat{a}-a_\bullet)\exp(iu\Delta\hat{b}(t))f(\hat{a}-a_\bullet)\rangle_{\rm eq}, \tag{IV.3}$$

where u is a real parameter. The expectation value and the variance of $\hat{b}(t)$ are obtained, respectively, by

$$\langle\hat{b}(t)\rangle_{a_\bullet} = -i\frac{\partial}{\partial u}\,\langle \exp(iu\Delta\hat{b}(t))\rangle_{a_\bullet}\Big|_{u=0}, \tag{IV.4}$$

$$\langle\Big(\hat{b}(t)-\langle\hat{b}(t)\rangle_{a_\bullet}\Big)^2\rangle_{a_\bullet} = -\frac{\partial^2}{\partial u^2}\,\ln\langle \exp(iu\Delta\hat{b}(t))\rangle_{a_\bullet}\Big|_{u=0}. \tag{IV.5}$$

Furthermore, by Theorems 5 and 6, we can interchange the limit and differentiation. Hence, by Theorem 4,

$$\begin{aligned}
&\langle \exp(iu\Delta\hat{b}(t))\rangle_{a_\bullet}\\
&= \frac{1}{p(a_\bullet)}\frac{1}{2\pi}\int d\xi d\xi' f[\xi]f[\xi']\langle e^{i\xi(\hat{a}-a_\bullet)}e^{iu\Delta\hat{b}(t)}e^{i\xi'(\hat{a}-a_\bullet)}\rangle_{\rm eq}.\\
&\xrightarrow[N\to\infty]{} \frac{1}{p(a_\bullet)}\frac{1}{2\pi}\int d\xi d\xi' f[\xi]f[\xi']\exp\big[-i(\xi+\xi')\Delta a_\bullet - \frac{1}{2}(\xi+\xi')^2\delta a_{\rm eq}^2\\
&\qquad\qquad - (\xi+\xi')u\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq} - \frac{1}{2}u^2\delta b(t)^2_{\rm eq}\\
&\qquad\qquad - (\xi-\xi')u\langle\frac{1}{2}[\hat{a},\hat{b}(t)]\rangle_{\rm eq}\big]\\
&= \frac{1}{p(a_\bullet)}\frac{1}{\sqrt{2\pi\delta a_{\rm eq}^2}}e^{-u^2\delta b(t)^2_{\rm eq}/2}\int dx f(x-u\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq})f(x+u\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq})\\
&\qquad\qquad\times\exp\big[-\frac{1}{2\delta a_{\rm eq}^2}(x+\Delta a_\bullet - iu\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq})^2\big] \qquad (IV.6)\\
&\equiv \frac{1}{p(a_\bullet)}e^{-u^2\delta b(t)^2_{\rm eq}/2}q(a_\bullet - iu\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq},\ u\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq}). \qquad (IV.7)
\end{aligned}$$

In the last line, we have defined the following function:

$$q(a'_\bullet, y) \equiv \frac{1}{\sqrt{2\pi\delta a_{\rm eq}^2}}\int dx f(x-y)f(x+y)\exp\big[-\frac{1}{2\delta a_{\rm eq}^2}(x+\Delta a'_\bullet)^2\big]. \tag{IV.8}$$

Note that $q(a'_\bullet, 0) = p(a'_\bullet)$ and $(\partial/\partial y)q(a'_\bullet, y)|_{y=0} = 0$.

By using these relations, we have

$$\begin{aligned}
\langle\hat{b}(t)\rangle_{a_\bullet} &= -i\,\frac{\partial}{\partial u}\ln\langle \exp(iu\Delta\hat{b}(t))\rangle_{a_\bullet}\Big|_{u=0}\\
&\xrightarrow[N\to\infty]{} -i\,\frac{\partial}{\partial u}\ln q(a_\bullet - iu\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq}, u\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq})\Big|_{u=0}\\
&= -\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq}\frac{\partial}{\partial a_\bullet}\ln q(a_\bullet,0) - i\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq}\,\frac{\partial}{\partial y}\ln q(a_\bullet,y)\Big|_{y=0}\\
&= -\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq}(\ln p)'. \qquad (IV.9)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\langle(\hat{b}(t)-\langle\hat{b}(t)\rangle_{a_\bullet})^2\rangle_{a_\bullet}\\
&=-\left.\frac{\partial^2}{\partial u^2}\ln\langle\exp(iu\Delta\hat{b}(t))\rangle_{a_\bullet}\right|_{u=0}\\
&\xrightarrow[N\to\infty]{}\delta b_{\rm eq}^2-\left.\frac{\partial^2}{\partial u^2}\ln q(a_\bullet-iu\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq},u\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq})\right|_{u=0}\\
&=\delta b_{\rm eq}^2+\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq}^2\frac{\partial^2}{\partial {a_\bullet}^2}\ln q(a_\bullet,0)\\
&\qquad+i\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq}\langle\frac{1}{2i}[\hat{a},\hat{b}(t)]\rangle_{\rm eq}\left.\frac{\partial}{\partial a_\bullet}\frac{\partial}{\partial y}\ln q(a_\bullet,y)\right|_{y=0}-\langle\frac{1}{2i}[\Delta\hat{a},\Delta\hat{b}(t)]\rangle_{\rm eq}^2\left.\frac{\partial^2}{\partial y^2}\ln q(a_\bullet,y)\right|_{y=0}\\
&=\delta b_{\rm eq}^2+\langle\frac{1}{2}\{\Delta\hat{a},\Delta\hat{b}(t)\}\rangle_{\rm eq}^2\frac{\partial^2}{\partial {a_\bullet}^2}\ln p(a_\bullet)-\langle\frac{1}{2i}[\Delta\hat{a},\Delta\hat{b}(t)]\rangle_{\rm eq}^2\left.\frac{\partial^2}{\partial y^2}\ln q(a_\bullet,y)\right|_{y=0}.
\end{aligned}
\tag{IV.10}
$$

C. Derivation of Eq. (16)

A sufficient condition on f_j for Eq. (16) is that $f_j\in L^2(\mathbb{R})$ and $\xi|f_j[\xi]|^2$ are absolutely integrable. This condition is satisfied if, for example, f_j are continuous and piecewise differentiable and f_j' (on each of their subdomains) are square integrable.

By Theorem 8, and $\int dx\ |f_j(x)|^2=1$ and $\int dx\ x|f_j(x)|^2=0$, and the spectrum decomposition of $f_j(\hat{a}^j-a_\bullet^j)$,

$$
\overline{\Delta a_\bullet^j}\equiv\int da_\bullet^0 da_\bullet^1\cdots da_\bullet^j\ \Delta a_\bullet^j\langle f_0(\hat{a}^0-a_\bullet^0)\cdots f_j(\hat{a}^j-a_\bullet^j)f_j(\hat{a}^j-a_\bullet^j)\cdots f_0(\hat{a}^M-a_\bullet^M)\rangle_{\rm eq}
\tag{IV.11}
$$

$$
\begin{aligned}
&=\int da_\bullet^0 da_\bullet^1\cdots da_\bullet^{j-1}dx\ \big\langle f_0(\hat{a}^0-a_\bullet^0)\cdots f_{j-1}(\hat{a}^{j-1}-a_\bullet^{j-1})f_j(x)\\
&\qquad\qquad\times\left(\Delta\hat{a}^j-x\right)f_j(x)f_{j-1}(\hat{a}^{j-1}-a_\bullet^{j-1})\cdots f_0(\hat{a}^M-a_\bullet^M)\big\rangle_{\rm eq}
\end{aligned}
\tag{IV.12}
$$

$$
\begin{aligned}
&=\int da_\bullet^0 da_\bullet^1\cdots da_\bullet^{j-1}\ \langle f_0(\hat{a}^0-a_\bullet^0)\cdots f_{j-1}(\hat{a}^{j-1}-a_\bullet^{j-1})\Delta\hat{a}^j f_{j-1}(\hat{a}^{j-1}-a_\bullet^{j-1})\cdots f_0(\hat{a}^M-a_\bullet^M)\rangle_{\rm eq}\\
&\xrightarrow[N\to\infty]{}\int d\xi_0\cdots d\xi_{j-1}|f_0[\xi_0]|^2\cdots|f_{j-1}[\xi_{j-1}]|^2\lim_{N\to\infty}\langle e^{i\xi_0\Delta\hat{a}^0}\cdots e^{i\xi_{j-1}\Delta\hat{a}^{j-1}}\Delta\hat{a}^j e^{-i\xi_{j-1}\Delta\hat{a}^{j-1}}\cdots e^{-i\xi_0\Delta\hat{a}^0}\rangle_{\rm eq}\\
&=\int d\xi_0\cdots d\xi_{j-1}|f_0[\xi_0]|^2\cdots|f_{j-1}[\xi_{j-1}]|^2 i\sum_{l=0}^{j-1}\xi_l\langle[\hat{a}^l,\hat{a}^j]\rangle_{\rm eq}\\
&=0.
\end{aligned}
\tag{IV.13}
$$

D. Derivation of Eq. (7) and Eq. (17)

A sufficient condition on f for Eqs. (7) and (17) is that $f_j\in L^2(\mathbb{R})$ and $f_j[\xi]$ are continuous and piecewise differentiable, and that $\xi f_j'[\xi]f_j[\xi]$ and $\xi^2|f_j[\xi]|^2$ are absolutely integrable. These conditions are satisfied if, for example, f_j are continuously differentiable and piecewise twice differentiable, and f_j, xf_j, f_j', f_j'' (on each of their subdomains) are square integrable.

We first prove Eq. (17). By Theorem 9, we have for $j < k$

$$
\begin{aligned}
&\overline{\Delta a^j_\bullet \Delta a^k_\bullet} \\
&\equiv \int da^0_\bullet da^1_\bullet \cdots da^k_\bullet \ \Delta a^j_\bullet \Delta a^k_\bullet \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_k(\hat{a}^k - a^k_\bullet) f_k(\hat{a}^k - a^k_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet)\rangle_{\rm eq} \qquad \text{(IV.14)} \\
&= \int da^0_\bullet da^1_\bullet \cdots da^{k-1}_\bullet \ \Delta a^j_\bullet \langle f_0(\hat{a}^0 - a^0_\bullet) \cdots f_{k-1}(\hat{a}^{k-1} - a^{k-1}_\bullet) \Delta \hat{a}^k f_{k-1}(\hat{a}^{k-1} - a^{k-1}_\bullet) \cdots f_0(\hat{a}^M - a^M_\bullet)\rangle_{\rm eq} \\
&\xrightarrow[N\to\infty]{} \int d\xi_0 \cdots d\xi_{k-1} |f_0[\xi_0]|^2 \cdots |f_{j-1}[\xi_{j-1}]|^2 \,{\rm Im}(f'_j[\xi_j] f^*_j[\xi_j]) |f_{j+1}[\xi_{j+1}]|^2 \cdots |f_{k-1}[\xi_{k-1}]|^2 \\
&\qquad\qquad \times \lim_{N\to\infty} \langle e^{i\xi_0 \Delta\hat{a}^0} \cdots e^{i\xi_{k-1}\Delta\hat{a}^{k-1}} \Delta\hat{a}^k e^{-i\xi_{k-1}\Delta\hat{a}^{k-1}} \cdots e^{-i\xi_0\Delta\hat{a}^0}\rangle_{\rm eq} \\
&\quad + \int d\xi_0 \cdots d\xi_{k-1} |f_0[\xi_0]|^2 \cdots |f_{k-1}[\xi_{k-1}]|^2 \\
&\qquad \times \lim_{N\to\infty} {\rm Re}\Big\langle e^{i\xi_0\Delta\hat{a}^0} \cdots e^{i\xi_{j-1}\Delta\hat{a}^{j-1}} e^{i\xi_j \Delta\hat{a}^j} \Delta\hat{a}^j e^{i\xi_{j+1}\Delta\hat{a}^{j+1}} \cdots e^{i\xi_{k-1}\Delta\hat{a}^{k-1}} \\
&\qquad\qquad\qquad \times \Delta\hat{a}^k e^{-i\xi_{k-1}\Delta\hat{a}^{k-1}} \cdots e^{-i\xi_0\Delta\hat{a}^0}\Big\rangle_{\rm eq}. \qquad \text{(IV.15)}
\end{aligned}
$$

By Theorem 3, we can evaluate these limits by the QCLT as

$$
\begin{aligned}
&\int d\xi_0 \cdots d\xi_{k-1} |f_0[\xi_0]|^2 \cdots |f_{j-1}[\xi_{j-1}]|^2 \,{\rm Im}(f'_j[\xi_j] f^*_j[\xi_j]) |f_{j+1}[\xi_{j+1}]|^2 \cdots |f_{k-1}[\xi_{k-1}]|^2 \\
&\qquad\qquad \lim_{N\to\infty} \langle e^{i\xi_0 \Delta\hat{a}^0} \cdots e^{i\xi_{k-1}\Delta\hat{a}^{k-1}} \Delta\hat{a}^k e^{-i\xi_{k-1}\Delta\hat{a}^{k-1}} \cdots e^{-i\xi_0\Delta\hat{a}^0}\rangle_{\rm eq} \\
&= -\int d\xi_0 \cdots d\xi_{k-1} |f_0[\xi_0]|^2 \cdots |f_{j-1}[\xi_{j-1}]|^2 \,{\rm Im}(f'_j[\xi_j] f^*_j[\xi_j]) |f_{j+1}[\xi_{j+1}]|^2 \cdots |f_{k-1}[\xi_{k-1}]|^2 \sum_{l=0}^{k-1} \xi_l \langle \frac{1}{i}[\hat{a}^l, \hat{a}^k]\rangle_{\rm eq} \\
&= -\int d\xi_j \,{\rm Im}(f'_j[\xi_j] f^*_j[\xi_j]) \xi_j \langle \frac{1}{i}[\hat{a}^j, \hat{a}^k]\rangle_{\rm eq} \\
&= \langle \frac{1}{i}[\hat{a}^j, \hat{a}^k]\rangle_{\rm eq} \,{\rm Im} \int dx x f_j(x) f'_j(x) \\
&= 0, \qquad \text{(IV.16)}
\end{aligned}
$$

$$
\begin{aligned}
&\int d\xi_0 \cdots d\xi_{k-1} |f_0[\xi_0]|^2 \cdots |f_{k-1}[\xi_{k-1}]|^2 \lim_{N\to\infty} {\rm Re}\Big\langle e^{i\xi_0\Delta\hat{a}^0} \cdots e^{i\xi_{j-1}\Delta\hat{a}^{j-1}} e^{i\xi_j\Delta\hat{a}^j} \Delta\hat{a}^j e^{i\xi_{j+1}\Delta\hat{a}^{j+1}} \cdots e^{i\xi_{k-1}\Delta\hat{a}^{k-1}} \\
&\qquad\qquad\qquad \Delta\hat{a}^k e^{-i\xi_{k-1}\Delta\hat{a}^{k-1}} \cdots e^{-i\xi_0\Delta\hat{a}^0}\Big\rangle \\
&= \int d\xi_0 \cdots d\xi_{k-1} |f_0[\xi_0]|^2 \cdots |f_{k-1}[\xi_{k-1}]|^2 \left(\langle \frac{1}{2}\{\hat{a}^j, \hat{a}^k\}\rangle_{\rm eq} + \sum_{l=0}^{k-1} \xi_l \langle \frac{1}{i}[\hat{a}^l, \hat{a}^k]\rangle_{\rm eq} \sum_{h=0}^{j-1} \xi_h \langle \frac{1}{i}[\hat{a}^h, \hat{a}^j]\rangle_{\rm eq} \right) \\
&= \langle \frac{1}{2}\{\hat{a}^j, \hat{a}^k\}\rangle_{\rm eq} + \sum_{l=0}^{j-1} \langle \frac{1}{i}[\hat{a}^l, \hat{a}^k]\rangle_{\rm eq}^2 \int d\xi_l \ \xi_l^2 |f_l[\xi_l]|^2 \\
&= \langle \frac{1}{2}\{\hat{a}^j, \hat{a}^k\}\rangle_{\rm eq} + \sum_{l=0}^{j-1} F_l \langle \frac{1}{2i}[\hat{a}^j, \hat{a}^l]\rangle_{\rm eq} \langle \frac{1}{2i}[\hat{a}^l, \hat{a}^k]\rangle_{\rm eq}. \qquad \text{(IV.17)}
\end{aligned}
$$

Here,

$$
F_l \equiv 4 \int dx \{f'_l(x)\}^2, \qquad \text{(IV.18)}
$$

where $f'_l(x)$ denotes the Fourier transform of $i\xi f_l[\xi]$. For the case where $f_l(x)$ is twice differentiable, F_l can be rewritten as $F_l = -4\int dx f''_l(x) f_l(x)$.

We finally prove Eq. (7) using Eq. (17). Putting $\hat{a}^0 = \hat{a}$ and $\hat{a}^1 = \hat{b}$, we have

$$\begin{aligned}
\overline{\Delta a^0_\bullet \Delta a^1_\bullet} &= \int da^0_\bullet da^1_\bullet \; \Delta a^0_\bullet \Delta a^1_\bullet \langle f_0(\hat{a}^0 - a^0_\bullet) f_1(\hat{a}^1(t_1) - a^1_\bullet) f_1(\hat{a}^1(t_1) - a^1_\bullet) f_0(\hat{a}^0 - a^0_\bullet)\rangle_{\rm eq} \\
&= \int da^0_\bullet dx \; \Delta a^0_\bullet \{f_1(x)\}^2 \langle f_0(\hat{a}^0 - a^0_\bullet)(\Delta \hat{a}^1(t_1) - x) f_0(\hat{a}^0 - a^0_\bullet)\rangle_{\rm eq} \\
&= \int da^0 \; \Delta a^0_\bullet \langle f_0(\hat{a}^0 - a^0_\bullet) \Delta \hat{a}^1(t_1) f_0(\hat{a}^0 - a^0_\bullet)\rangle_{\rm eq} \\
&= \overline{\Delta a^0_\bullet \langle \Delta \hat{a}^1(t_1)\rangle_{a^0_\bullet}}. \qquad \text{(IV.19)}
\end{aligned}$$

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