Quantum kicked rotors and related systems

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1 Introduction

A quantum kicked rotor (QKR) is one of the origins of *quantum chaos*, which roughly implies the field of study on *classically chaotic* quantum systems. Since the first numerical investigation of a QKR by Casati *et al.* [1], there have appeared many descendants of it. In order to settle the situation, I here review QKRs and the related systems which include a classical kicked rotor, an interacting boson system, an electron-phonon system and others.

2 A classical kicked rotor (CKR)

First of all, I discuss the properties of a *classical* kicked rotor (CKR). The Hamiltonian for a CKR is,

$$\boxed{H(t) = p^2/2I + kV(\theta)\sum_n \delta(t-n),} \tag{1}$$

where p denotes an angular momentum, I a moment of inertia, K the strength of the nonlinearity, and the period of the kick is normalized to be one. Note that this is a (periodically) *time-dependent* Hamiltonian, and there is no *stochastic (noise) term* in it. The equations of motion for a CKR *when* $V(\theta) = \cos\theta$, after some normalizations of the variables, are

$$\theta_{n+1} = (\theta_n + p_n) \mod 2\pi, \tag{2}$$

$$p_{n+1} = p_n + k\sin\theta_{n+1}, \tag{3}$$

where n represents the discretized time step. This is called *the standard map* [2, 3]. Due to the delta function in Eq. (1), it *exactly* becomes a discrete map. So the numerical calculation is efficiently done. Iterating the map, we can calculate, for example, the dynamics of the variance of p, and it was found that, under some conditions,

$$\boxed{\langle (p_n - p_0)^2 \rangle \simeq Dn,} \tag{4}$$

where the brackets mean the average for the initial ensemble, and $D \simeq k^2/2$, which is numerically known. This behavior is very similar to that of *diffusion* [cf. $\langle (x_t - x_0)^2 \rangle \simeq D't$], so this is called "the diffusion of momentum". This issue, which is understood as an example of the collapse of KAM tori and the *spontaneous* emergence of stochasticity, was intensively studied by Chirikov and his coworkers [4].

3 A quantum kicked rotor (QKR)

It is natural to ask what happens when one quantizes the CKR. This was first numerically investigated by Casati *et al.* [1]. They employed the following Schrödinger equation for a *quantum* kicked rotor (QKR),

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\theta,t) = H(t)\Psi(\theta,t) = \left[-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} + kV(\theta)\sum_n \delta(t-n)\right]\Psi(\theta,t),} \tag{5}$$

where $H(t)$ is given by Eq. (1) with the replacement $p = (\hbar/i)\partial/\partial\theta$. Then, they numerically calculated the wavefunction of a QKR from Eq. (5) as follows:

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- First, take the initial wave function. [e.g., $\Psi(\theta, t=0) = (2\pi)^{-1/2}$.]
- Second, multiply the wave function by $\exp[-i(k/\hbar)V(\theta)]$.
- Fourier transform it $[\Psi(\theta) \to \Psi_l]$.
- Multiply the Fourier transformed wave function by $\exp[-i\hbar l^2/2I]$.
- Inverse Fourier transform it $[\Psi_l \to \Psi(\theta)]$.
- Return to the second step.

Iterating this loop, we obtain the dynamics of the wavefunction $\Psi(\theta, t)$, and can calculate the quantum average of any physical quantity. To discuss the relation between a CKR and a QKR, they calculated the variance of p as follows,

$$\langle p_t^2 \rangle = \hbar^2 \sum_{l=-\infty}^{\infty} l^2 |\Psi_l(t)|^2 \tag{6}$$

where the brackets mean the quantum average, and we have assumed, for simplicity, $\langle p_0 \rangle = 0$, i.e., $\Psi(\theta, 0) = (2\pi)^{-1/2}$. *When* $V(\theta) = \cos\theta$, they found, contrary to the case of a CKR, that a QKR stops diffusing [1] in momentum space at a certain time t^*. This result, *suppression of chaos in a quantum system*, seems to lead, in general, to the absence of the *quantum stochasticity* [2, 6], which denotes such concepts as mixing property or approaches to equilibrium states in quantum systems. However, there are some ways to recover the stochasticity in quantum systems as discussed later.

4 Mapping a QKR onto a tight-binding model

Mapping is a useful concept when we understand something deeper. In this section, following Fishman *et al.* [7], we map *one* QKR onto another problem, which is related to 1D Anderson Localization (AL) [8]. Since the issue on AL has been well studied in solid-state physics [9], we can, to understand QKRs, use the method developed in the AL literature, and vice versa.

In the following, I review the procedure in detail: The Hamiltonian in Eq. (5) has the periodicity, $H(t+1) = H(t)$, so the Floquet's theorem can be applied, i.e., the wavefunction can be written as $\Psi(\theta, t) = e^{-i\omega t} u_\omega(\theta, t)$, where $u_\omega(\theta, t+1) = u_\omega(\theta, t)$ and ω is called a *quasi-energy* (which is like a Bloch index in the field of solid-state physics [10]). Hence, all we need is the dynamics from $t = 0_+$ (just after the kick at $t = 0$) to $t = 1$. Since the particle freely moves between $t = 0_+$ and $t = 1_-$ (just before the kick at $t = 1$), the transformation is

$$u_{-,m} = u_{+,m} e^{iE_m}, \tag{7}$$

where $u_+(\theta) = u_+(\theta, t = 0_+)$, $u_-(\theta) = u_-(\theta, t = 1_-)$, $E_m = \omega - \tau m^2/2$, $\tau = 1/I$, and we have defined the Fourier component of $f(\theta)$, f_m, as:

$$f(\theta) = \sum_m f_m e^{im\theta}, \qquad f_m = \frac{1}{2\pi} \int_0^{2\pi} d\theta f(\theta) e^{-im\theta}.$$

Inverse Fourier transforming Eq. (7), we obtain

$$u_-(\theta) = \sum_m \int \frac{d\phi}{2\pi} e^{im(\theta-\phi)+iE_m} u_+(\phi). \tag{8}$$

Between $t = 1_-$ and $t = 1_+ = 0_+$, the particle feels a kick and is transformed to be

$$u_+(\theta) = u_-(\theta) e^{-ikV(\theta)}. \tag{9}$$

Next, we define $\bar{u} = (u_+ + u_-)/2$, and have, from Eq. (9), $\bar{u} = u_+/(1 - iU) = u_-/(1 + iU)$, where $U = -\tan[kV(\theta)/2]$. Substituting these into Eq. (8) to eliminate u_+, u_-, we have

$$\begin{aligned}
[1 - iU(\theta)]\bar{u}(\theta) &= \sum_m \int \frac{d\phi}{2\pi} e^{im(\theta-\phi)+iE_m} [1 + iU(\phi)]\bar{u}(\phi) \\
&= \sum_m e^{i(E_m+m\theta)} \int \frac{d\phi}{2\pi} \left[\sum_r u_r e^{i(r-m)\phi} + i \sum_n \sum_r U_n u_r e^{i(n+r-m)\phi} \right] \\
&= \sum_m e^{i(E_m+m\theta)} \left[u_m + i \sum_r U_{m-r} u_r \right].
\end{aligned}$$

[1] As Izraelev and Shepelyansky showed [5], for special values of $\hbar$, there is the cases where $\langle p_t^2 \rangle \propto t^2$. This is called "quantum resonance".

Fourier transforming the above equation, we obtain the left-hand side (l.h.s.) and right-hand side (r.h.s.) of the equation as

$$\begin{aligned} \text{l.h.s.} &= u_n - i\sum_r U_{n-r}u_r, \\ \text{r.h.s.} &= e^{iE_n}(u_n + i\sum_r U_{n-r}u_r). \end{aligned}$$

Equating these, we finally have

$$\boxed{T_n u_n + \sum_r U_{n-r}u_r = 0,} \tag{10}$$

where $T_n = \tan(E_n/2)$. This is just the tight-binding model in solid-state physics [10], if we regard the subscript "r" of u_r as a "site" in *momentum* space, T_n as a site energy, and U_r as a hopping energy. In the AL literature, the site energy T_n is a stochastic variable, whereas it is just a *deterministic function* of n here. In spite of this fact, if τ is an *irrational number*, $T_n = \tan(E_n/2)$ behaves *like random*, [2] and AL *does* appear even if Eq. (10) is employed! This is a very interesting and striking result that Fishman *et al.* showed in their numerical calculations.

On the other hand, if τ is a rational number, T_n becomes a periodic function of n. Hence, all the states become extended (Bloch) ones, in other words, AL disappears. They considered that this case is rather rare in physics, and that AL always will appear. However, because they did not find an *exact* link between a QKR and 1D AL, their argument, in a sense, was not a complete one. Recently, Altland and Zirnbauer [14] showed that the behaviors of a QKR *in low-frequency regions* can be mapped onto a supersymmetric nonlinear sigma (SUSYNS) model. On the other hand, it is known that AL problem *without electron-electron interactions* also can be mapped onto the same model [15]. Therefore, the relation, 1D AL $\sim$ a SUSYNS model $\sim$ a QKR was established. Strictly speaking, however, their argument only concerned with a link between *one* QKR and 1D AL. What happens when we extend one-body QKR to many-body (or higher dimensional) ones ?

5 Extension to many-body QKR systems

There are many ways to extend a QKR to many-body ones. One possibility is the following Hamiltonian,

$$\boxed{H(t) = -\frac{\hbar^2}{2I}\sum_i \frac{\partial^2}{\partial\theta_i^2} + kV(\{\theta_k\})\sum_n \delta(t-n),} \tag{11}$$

where $V(\{\theta_k\}) = V(\theta_1, \theta_2, ..., \theta_N)$ can be complex. In particular, the following form of the potential energy,

$$\boxed{V(\{\theta_k\}) = \sum_{i=1}^{N} K_i \cos\theta_i + \sum_{i\neq j}^{N} \epsilon_{ij}\cos(\theta_i - \theta_j),} \tag{12}$$

was examined by Toda, Adachi, and Ikeda [16]. In a physical sense, this is a natural choice because it resembles a Heisenberg XY model in the field of magnetism. They showed in their numerical calculations that even if N (the total number of QKRs) $= 2$, there appears, in a sense, again the diffusion of the momentum which exists in a CKR.

Now let us map this ($N = 2$-QKR) problem onto 2D AL. This is a generalization of the method by Fishman *et al.*:

$$T_{\mathbf{n}}u_{\mathbf{n}} + \sum_{\mathbf{r}} U_{\mathbf{n}-\mathbf{r}}u_{\mathbf{r}} = 0, \tag{13}$$

where $\mathbf{n} = (n, m)$ is a *pair* of integers. If we use the analogy with 2D AL, there should be localization due to the scaling argument [17]. According to Toda *et al.*, however, *there is not !* This remarkable discrepancy, I think, deserves further considerations. **We need to calculate the localization length and/or something else in 2D (or higher) momentum space**, like Shepelyansky did in the 1D space [18]. We also must investigate

[2] This is also called *quasi-random* (or *pseudo-random*). In fact, since, in numerical calculations, some algorithms must be used to generate random numbers, *only* quasi-random numbers are available for us. As Kolmogorov, Chaitin, and Solomonoff independently showed, we, *in principle*, can *not* judge which number is *exactly* random though we *can* define randomness [11]. This fact is, as the halting problem of a (universal) Turing machine [12], another version of Gödel's incompleteness theorem [13].

how the property of the hopping integral $U_{\mathbf{n}-\mathbf{r}}$ affects the result. In the textbook, it is often assumed that $U_{\mathbf{n}-\mathbf{r}}$ has only nearest-neighbor components, but it is not the case in general. [3]

Furthermore, it is known that a 3D-random system exhibits the metal-insulator (MI) transition. So, **what is the correspondence of the MI transition in quantum chaos ?** This issue was considered by Casati *et al.* [19] in the following many-body QKR model,

$$H(t) = H_0(n) + \omega_1 n_1 + \omega_2 n_2 + V(\theta, \theta_1, \theta_2) \sum_n \delta(t-n), \tag{14}$$

where $H_0(n)$ is a random Hamiltonian (this might be a trick !), $n_i = -i\partial/\partial\theta_i$ (i=1,2), and ω_1, ω_2 incommensurate frequencies. There are three degrees of freedom, $\theta, \theta_1, \theta_2$, so we can map this onto 3D AL. They, in particular, took the following potential,

$$V(\theta, \theta_1, \theta_2) = -2\tan^{-1}[2k(\cos\theta + \cos\theta_1 + \cos\theta_2)],$$

for the mapped model to have *only* nearest-neighbor hopping. They then numerically calculated the localization length in the case of localization, and the diffusion of momentum in the case of delocalization, and showed that there occurs the localization-delocalization transition corresponding to the MI transition in 3D-random systems.

There exist other generalizations of a QKR [20, 21]. Since the number of the generalizations is, of course, enormous, **we need some phenomenology to classify many-body QKR systems** for economic reasons. This issue remains to be investigated.

6 Many-body quantum systems except QKRs

Next, we will consider some many-body systems except QKRs. The first example is the *quantum hard sphere gas* [22], which is described by the following Schrödinger equation,

$$-\frac{\hbar^2}{2m}\sum_a \nabla_a^2 \Psi = E\Psi, \tag{15}$$

where $\Psi = 0$ at the boundary defined by

$$|\mathbf{r}_a - \mathbf{r}_b| \geq d, \quad d/2 \leq x_a, y_a, z_a \leq L - d/2.$$

Its classical counterpart was studied by Krylov and Sinai [23], who showed that the classical hard sphere gas has positive *Kolmogorov-Sinai entropy*, which is one of the indicators of stochasticity in classical physics. For the semiclassical treatment of quantum hard sphere gas, see Ref. [22].

The second is the interacting boson system studied by Jona-Lasinio and Presilla [24]. Their Hamiltonian is

$$H = \sum_{j=1}^{L}[\alpha_j a_j^\dagger a_j - \beta_j(e^{i\theta}a_{j+1}^\dagger a_j + \text{h.c.})] + \sum_{j=1}^{L}\gamma_j a_j^\dagger a_j^\dagger a_j a_j, \tag{16}$$

where the operator $a_j^\dagger$ creates a boson in the site j, and α_j, β_j and γ_j are the site, hopping, and interaction energies, respectively. They numerically calculated the following quantity,

$$\kappa_{AB} = \lim_{t\to\infty}\sqrt{\frac{1}{t}\int_0^t dt' |\langle A(t)B\rangle - \langle A\rangle\langle B\rangle|}, \tag{17}$$

which indicates the potential of a mixing property, i.e., the maximal mixing implies that $\kappa_{AB} = 0$. They showed that as the number of the particles increases (although the calculated number is only up to seven), κ_{AB} decreases to zero. This result agrees with our intuition. They also studied its classical counterpart with mean-field approximation. The approximated equation of motion derived from Eq. (16) is

$$i\hbar\frac{d}{dt}z_j(t) = [\alpha_j + 2(N-1)\gamma_j|z_j(t)|^2]z_j(t) - \beta_{j-1}e^{i\theta}z_{j-1}(t) - \beta_j e^{-i\theta}z_{j+1}(t), \tag{18}$$

where $z_j(t)$ is the amplitude of the mean field in the site j. They numerically showed that when the classical system described by Eq. (18) becomes chaotic *in a classical sense*, the corresponding quantum system described by Eq. (16) has a mixing property. This is one of the examples of *quantum-classical correspondence* studied

[3] Of course, after renormalization procedure, there might appear a universality where *only* nearest-neighbor hopping is relevant.

in the field of quantum chaos. Similar *spin* systems were also treated by Jensen, Shankar and others [25], and it was shown that even *closed small* integrable (and, of course, non-integrable) systems *numerically* exhibit a mixing property.

The third is the 1D-random electron system coupled with *a few modes* of phonon, which was studied by Yamada and Ikeda [26]. Their Hamiltonian is

$$\begin{aligned} H &= H_{\rm el} + H_{\rm ph} + H_{\rm int}, \\ &= \sum_{n=1}^{N} |n\rangle V(n)\langle n| + \sum_{n\neq m}^{N} |n\rangle K(n,m)\langle m| \\ &\quad + \sum_{j=1}^{M}(\frac{1}{2}\hat{p}_j^2 + \frac{1}{2}\omega_j^2\hat{q}_j^2) + \sum_{n=1}^{N}\sum_{j=1}^{M}[|n\rangle V(n)\langle n|]b_j\hat{q}_j, \end{aligned} \tag{19}$$

where $V(n)$ is the random site energy distributed in $[-W,W]$, $K(n,m)$ the hopping energy, $\hat{q}_j, \hat{p}_j$ the coordinate and momentum of phonons, respectively, b_j the coupling strength between the electron and phonons. (A similar, but less quantum, model was examined by Kopidakis *et al.* [27]. One may point out the similarity between Eq. (14) and above.) They numerically calculated from Eq. (19) the mean-square displacement,

$$\langle \Delta\hat{n}^2\rangle = \langle\Psi(t)|(\hat{n} - \langle\hat{n}\rangle)^2|\Psi(t)\rangle,$$

where $\hat{n} = \sum_{j=1}^{N} j|j\rangle\langle j|$ is the position operator and $|\Psi(t)\rangle$ is the time-dependent wave function. This quantity characterizes whether the electron becomes localized or extended. Their numerical experiments showed that there is an anomalous diffusion law, $\langle\Delta\hat{n}^2\rangle = Dt^{\alpha}$, and α approaches one from below as the number of the phonon mode increases. (When the number is *three*, we can conceive normal diffusion: $\langle\Delta\hat{n}^2\rangle \simeq Dt$.) At the same time, they calculated the energy of the electron system, $E_{\rm el}(t) = \langle\Psi(t)|H_{\rm el}|\Psi(t)\rangle$, and that of the phonon system, $E_{\rm ph}(t) = \langle\Psi(t)|H_{\rm ph}|\Psi(t)\rangle$. They showed, when the coupling b_j and the randomness W are strong enough, there exists "a one-way quasi-stationary energy flow from electron to phonons" [26].

Next, we see, in Yamada-Ikeda model, what happens when all the phonon modes are excited in coherent states, i.e., all the phonons are *classicalized*. [4] The procedure is as follows: Unitary transform the Schrödinger equation $i\hbar\frac{\partial}{\partial t}\Psi(t) = (H_{\rm el} + H_{\rm ph} + H_{\rm int})\Psi(t)$ into another one for $\phi(t) = \exp(iH_{\rm ph}t/\hbar)\Psi(t)$, which satisfies,

$$i\hbar\frac{\partial}{\partial t}\phi(t) = [H_{\rm el} + e^{iH_{\rm ph}t/\hbar}H_{\rm int}e^{-iH_{\rm ph}t/\hbar}]\phi(t).$$

We can expand $e^{iH_{\rm ph}t/\hbar}H_{\rm int}e^{-iH_{\rm ph}t/\hbar}$ as

$$\begin{aligned} e^{iH_{\rm ph}t/\hbar}H_{\rm int}e^{-iH_{\rm ph}t/\hbar} &= \sum_{n=1}^{N}\sum_{j=1}^{M}[|n\rangle V(n)\langle n|]b_j\sqrt{\frac{\hbar}{2\omega_j}}(\hat{a}_j e^{-i\omega_j t} + \text{h.c.}) \\ &\simeq \sum_{n=1}^{N}\sum_{j=1}^{M}[|n\rangle V(n)\langle n|]\epsilon_j\cos(\omega_j t + \phi_j), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \epsilon_j &= \sqrt{2\hbar b_j^2|\alpha_j|^2/\omega_j}, \quad \hat{q}_j = \sqrt{\hbar/2\omega_j}(\hat{a}_j + \hat{a}_j^{\dagger}), \\ H_{\rm ph} &= \sum_{j=1}^{M}\hbar\omega_j(\hat{a}_j^{\dagger}\hat{a}_j + 1/2), \quad e^{iH_{\rm ph}t/\hbar}\hat{a}_j e^{-iH_{\rm ph}t/\hbar} = \hat{a}_j e^{-i\omega_j t}, \end{aligned}$$

and we have used the following *approximation*,

$$\hat{a}_j \simeq |\alpha_j|e^{-i\phi_j},$$

(where α_j, ϕ_j are c-numbers) which is justified when all the phonons are *classicalized* due to decoherence or something. Hence, the reduced system is described by

$$H'_{\rm el} = \sum_{n=1}^{N}|n\rangle V(n,t)\langle n| + \sum_{n\neq m}^{N}|n\rangle K(n,m)\langle m|,$$

[4] They instead considered the case when all the phonon modes are excited around *Fock state* with a large quantum number [26]. But, this, I think, is a rather unphysical choice, because the states are highly nonclassical.

where

$$V(n,t) = V(n)[1 + \sum_{j=1}^{M} \epsilon_j \cos(\omega_j t + \phi_j)].$$

This system, with incommensurate frequencies ω_j, was also investigated by them and Goda [28]. It was shown that the periodic perturbation causes the destruction of localization. A little different system in *more than two dimensions* is being examined by Brandes [29].

The systems discussed above are only a few examples of many-body quantum chaos systems. There will be, of course, other interesting systems to study the quantum stochasticity.

7 Experimental aspects of quantum chaos systems

In this section, I discuss the experimental aspects of quantum chaos systems. From a historical point of view, the results of experiments [30] on ionization of Rydberg atoms by an external field were first shown to have the properties of a quantum chaos system [31]. The Hamiltonian considered is

$$H_0(t) = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{x} + exF\sin(\omega t), \quad x > 0, \tag{21}$$

where F and ω are the field strength and frequency in atomic units. This is a model rather different from a QKR, but it has similar consequences, including quantum limitation of chaotic excitation (ionization). In real situations, of course, *noise* affects the behaviors of the system considered. This issue was investigated by Blümel *et al.* [32]. Their total Hamiltonian, including the variables for the environment, is

$$H(t) = H_0(t) + H_{\rm int} + H_{\rm R},$$

with

$$H_{\rm int} = \hbar \sum_i g(\omega_i) x (b_i + b_i^\dagger), \quad H_{\rm R} = \hbar \sum_i \omega_i (b_i^\dagger b_i + 1/2).$$

Their central result is that small amount of noise causes the destruction of quantum coherence and the localization in energy space, which is the origin of quantum limitation of chaotic excitation, disappears. This is very similar to the results in the case of a QKR [33]. In solid-state physics literature, this is similar to those by Thouless [34] who considered the effect of finite temperature on localization in a solid.

Those discussed above are not directly related to QKRs. But, in the field of atom optics, there exists an ideal realization of a QKR [35]. The situation is as follows: An atom is exposed to the delta-pulse trains of light, which simultaneously simulate the potential and the delta functions in Eq. (5). It was shown that the atom-optics system of a QKR *experimentally* exhibits the same behavior as predicted by Casati *et al.* [1].

Atom-optics systems are now being studied in connection with Bose-Einstein condensation. I think **these systems are also promising, both experimentally and theoretically, to investigate quantum chaos sytems, including many-body QKR ones, and to do quantum stochasticity**.

8 Summary

Quantum kicked rotors (QKRs) are typical examples in the field of quantum chaos. A QKR causes the suppression of chaos in very short time scale, and that is all. However, if there are more degrees of freedom, many things can happen. We need more calculations for many-body QKR systems, and some phenomenology to classify them. In order to achieve these, it is important to map them onto other systems (including tight-binding models, a supersymmetric nonlinear sigma model, and many others), because the crossreference which occurs there makes our understanding deeper. Atom-optics systems seem to be experimentally best to test the theoretical calculations on QKRs. Of course, except QKRs, there are many other quantum chaos systems, and many things remain to be done.

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