Breaking of Gauge Symmetry in Finite Systems

Akira Shimizu

Department of Basic Science, University of Tokyo, Komaba, Tokyo

Collaborator: Takayuki Miyadera （現：産総研）

Phase transition:

Theory is well developed for infinite systems.

But, real systems are finite. Not sufficiently understood.

This talk: Breaking of the U(1) gauge symmetry in finite systems.

Bose-Einstein condensate, Superconductor, etc.

- Finite $\longrightarrow$ superselection rule
- Macroscopic (Thermodynamical) $\longrightarrow$ stability

What is the vaccuum (or equilibrium) state that is compatible with the superselection rule and the macroscopic stability?

Superselection rule — popular, but misleading versions

- 電荷の異なる状態の重ね合わせは存在しない
- 電荷の異なる状態を重ね合わてはいけない

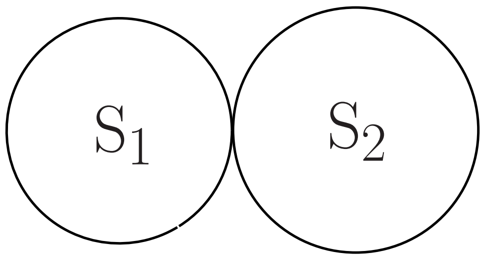
Popular arguments

Bose-Einstein condensate (BEC)やSuperconductorでは、

- 秩序変数$\hat{\mathcal{O}}$は、ゲージ不変でない：
  - BEC: $\hat{\mathcal{O}}(x) = \hat{\phi}(x)$
  - Superconductor : $\hat{\mathcal{O}}(x) = \hat{\psi}_\uparrow(x)\hat{\psi}_\downarrow(x)$
- superselection ruleにより、Nの定まった状態しか許されない.
- ゆえに$\langle\hat{\mathcal{O}}(x)\rangle = 0$であり、spontaneous symmetry breakingはない.
- しかし、Long-range orderはある：

$$\langle\hat{\mathcal{O}}^\dagger(x)\hat{\mathcal{O}}(y)\rangle \not\to 0 \text{ as } |x-y| \to \infty \quad \text{spatially.}$$

- Definite relative phase only when S_1 and S_2 are entangled.



$$J \propto \sin(\theta_1 - \theta_2)$$

- ゆえに、$\langle\hat{\mathcal{O}}(x)\rangle \neq 0$やcoherent stateを仮定した議論は正しくない.

Questions about the popular arguments

- ?? superselection rule により、N の定まった状態しか許されない.
- ?? ゆえに $\langle\hat{\mathcal{O}}(x)\rangle = 0$ であり、spontaneous symmetry breaking はない.
- ?? Definite relative phase only when S_1 and S_2 are entangled.
- ?? ゆえに、$\langle\hat{\mathcal{O}}(x)\rangle \neq 0$ や coherent state を仮定した議論は正しくない.

Moreover,

量子論で様々な状態が可能なとき、macroscopic (thermodynamical) stability がある状態を採用すべき！

N の定まった状態では、macroscopic stability がないのでは？

孤立有限系 vs 表面を通して外界と相互作用する有限系
→熱力学では、どちらも同じ平衡状態が実現すると仮定している
→もしも量子論で両者が異なるならば、熱力学と整合するのは後者！

CONTENTS

- Wrong points in the populer arguments.
- Then, what state is realized?
- Time evolution of $|vac\rangle$.
- Summary and conclusions.

Superselection rule (SSR) — a more precise description

清水明「新版 量子論の基礎」(サイエンス社, 2004) p.68

ある場合には，純粋状態の重ね合わせが，混合状態になることがある．これを，**超選択則**がある，と言う．

例えば，$|\psi_1\rangle$ = 電子が１個ある状態，$|\psi_2\rangle$ =電子が２個ある状態とすると，ゲージ不変な演算子 $\hat{A}$ については，必ず $\langle\psi_1|\hat{A}|\psi_2\rangle = 0$ となる．

そのため，ゲージ不変なものだけが可観測量となる* ような状況では，$|\psi_1\rangle$ と $|\psi_2\rangle$ を重ね合わせた状態は，全ての可観測量に対して干渉項が消えてしまい，混合状態になる．

「超選択則」という名前から，「そのような重ね合わせが禁止される」と早とちりされがちであるが，そうではなくて，「重ね合わせても良いのだけれど，$\langle\psi_1|\hat{A}|\psi_2\rangle \neq 0$ であるような可観測量がなければ，混合状態になりますよ」ということである．

---

*) これは，測定結果がゲージ不変であるべし，という物理的要求を満たすための**十分**条件である．

Definition of pure and mixed states

Let $\omega(A)$ represent the expectation value of an observable A in a state ω.

Def. ω is called mixed iff there exist ω_1 and ω_2 ($\neq \omega_1$) s.t.

$$\omega(A) = \lambda\, \omega_1(A) + (1-\lambda)\, \omega_2(A) \quad (0 < \lambda < 1)$$

for *every observable* A. Otherwise, ω is called pure.

Valid for both quantum and classical states.

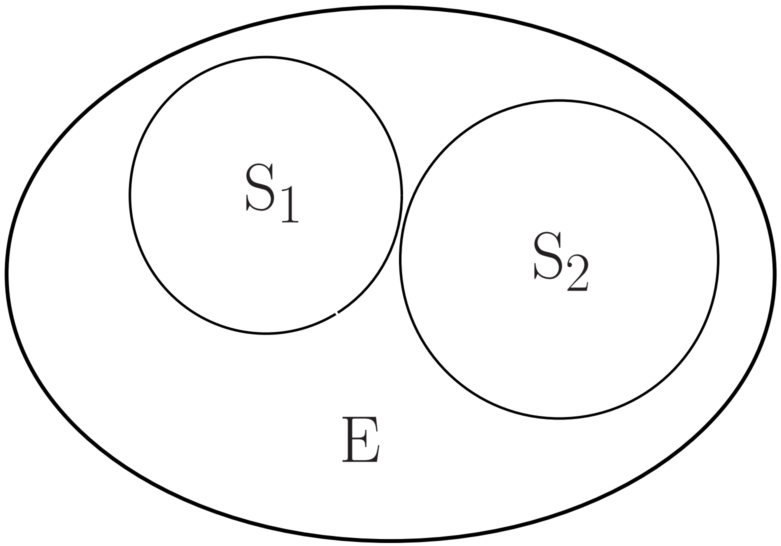
Note: The popular definition is

$$\hat{\rho}^2 \neq \hat{\rho} : \text{ mixed}, \; \hat{\rho}^2 = \hat{\rho} : \text{ pure}.$$

However, this is rather misleading:

- What observables are considered? (Are they gauge invariant? etc)
- What is the limit of $V \to \infty$?
- A mixed phase or mixed state in a pure phase?
- On what space $\hat{\rho}$ is defined?

An isolated quantum system that is subject to the SSR



- Look at $S \equiv S_1 + S_2$, and regard the rest as the environment E.
- [size of E] $\gg$ [size of S].
- One will not measure observables in E.
- S_2 can be (a part of) an apparatus for measuring S_1.

$$\text{Hilbert space}: \ \mathcal{H}_{\rm tot} = \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_2}_{\mathcal{H}_{\rm S}} \otimes \mathcal{H}_{\rm E}$$

$$\text{Total charge}: \ \hat{N}_{\rm tot} = \underbrace{\hat{N}_1 + \hat{N}_2}_{\hat{N}_{\rm S}} + \hat{N}_{\rm E}$$

$$|N_k \, \ell\rangle_k \equiv \ \text{eigenfunction of } \hat{N}_k \quad (k = 1, 2, \text{E})$$

We show ... AS and T. Miyadera, cond-mat/0102429.

There exist eigenstates $|\Phi\rangle_{\rm tot}$ of $\hat{N}_{\rm tot}$ with the following properties:

(i) The density operator $\hat{\rho}_{\rm S} \equiv {\rm Tr}_{\rm E}\left(|\Phi\rangle_{\rm tot\ tot}\langle\Phi|\right)$ satisfies $\hat{\rho}_{\rm S}^2 \neq \hat{\rho}_{\rm S}$.
But, $\hat{\rho}_{\rm S}$ is equivalent to a vector state $|\Phi_{\rm S}\rangle_{\rm S}$ ($\in \mathcal{H}_{\rm S}$) for any gauge-invariant observables in S.

(ii) $|\Phi_{\rm S}\rangle_{\rm S}$ is a product of vector states of $\rm S_1$ and $\rm S_2$;

$$|\Phi_{\rm S}\rangle_{\rm S} = |\Phi^{(1)}\rangle_1 |\Phi^{(2)}\rangle_2,$$

where $|\Phi^{(k)}\rangle_k$ is superposition of states with different charges,

$$|\Phi^{(k)}\rangle_k = \sum_{N_k,\,\ell} C^{(k)}_{N_k \ell} |N_k \ell\rangle_k.$$

→ $\rm S_1$ and $\rm S_2$ are not entangled, but can have definite relative phase!

(iii) To each subsystem $\rm S_k$, one can associate $|\Phi^{(k)}\rangle_k$ and observables which are *not* necessarily gauge invariant in each subsystem.

(iv) In this association, $|\Phi^{(k)}\rangle_k$ is a pure state.

Proof: a state with such properties

$$|\Phi\rangle_{\rm tot} = \sum_{N_1,\ell_1}\sum_{N_2,\ell_2}\sum_{\ell} C^{(1)}_{N_1\ell_1} C^{(2)}_{N_2\ell_2} C^{(\rm E)}_{N_1+N_2\ell} |N_1\ell_1\rangle_1 |N_2\ell_2\rangle_2 |N_{\rm tot} - N_1 - N_2, \ell\rangle_{\rm E}$$

- $\displaystyle\sum_{N_1,\ell_1} \left|C^{(1)}_{N_1\ell_1}\right|^2 = \sum_{N_2,\ell_2} \left|C^{(2)}_{N_2\ell_2}\right|^2 = \sum_{\ell} \left|C^{(\rm E)}_{N_1+N_2\ell}\right|^2 = 1.$
- $C^{(1)}_{N_1\ell_1} C^{(2)}_{N_2\ell_2}$ are non-vanishing only when $N_1 + N_2 \ll N_{\rm tot}$.
  States with low energies would satisfy this condition.

Therefore,

- $|\Phi\rangle_{\rm tot}$ is an eigenstate of $\hat{N}_{\rm tot}$.
- For $N_{\rm S} = N_1 + N_2$,

$$\text{Prob}[N_{\rm tot} - N_{\rm S} \text{ bosons in E}] \simeq \text{independent of } N_{\rm S}$$

$$\text{when } |N_{\rm S} - \langle N_{\rm S}\rangle| \ll \sqrt{\langle \delta N_{\rm S}^2\rangle}.$$

Natural for a large environment.

Reduced density operator of S (=$\mathbf{S}_1$+$\mathbf{S}_1$)

For a state $|\Phi\rangle_{\rm tot}$ of the total system,

$$\begin{aligned}\hat\rho_{\rm S} &= {\rm Tr}_{\rm E}\left(|\Phi\rangle_{\rm tot}\ {}_{\rm tot}\langle\Phi|\right)\\ &= \sum_{N_1',\ell_1'}\sum_{N_2',\ell_2'}\sum_{N_1,\ell_1}\sum_{N_2,\ell_2}\delta_{N_1+N_2,\ N_1'+N_2'}\\ &\qquad\times C^{(1)}_{N_1'\ell_1'}C^{(2)}_{N_2'\ell_2'}C^{(2)*}_{N_2\ell_2}C^{(1)*}_{N_1\ell_1}\ |N_1'\ell_1'\rangle_1|N_2'\ell_2'\rangle_2\ {}_2\langle N_2\ell_2|_1\langle N_1\ell_1|\end{aligned}$$

Except for the trivial case where $\sum_\ell |C^{(k)}_{N\ell}|^2 = \delta_{N,N_0^{(k)}}$, we find

$$(\hat\rho_{\rm S})^2 \neq \hat\rho_{\rm S} \quad \rightarrow \text{ `mixed state.'}$$

But, $(\hat\rho_{\rm S})^2 \neq \hat\rho_{\rm S}$ only ensures that for any vector state $|\Phi\rangle_{\rm S}$ ($\in \mathcal{H}_{\rm S}$) there exists some *operator* $\hat\Xi_{\rm S}$ (on $\mathcal{H}_{\rm S}$) for which

$${\rm Tr}_{\rm S}\left(\hat\rho_{\rm S}\hat\Xi_{\rm S}\right) \neq {}_{\rm S}\langle\Phi|\hat\Xi_{\rm S}|\Phi\rangle_{\rm S}.$$

Such $\hat\Xi_{\rm S}$ is not necessarily gauge-invariant, hence might not be an *observable* of S.

Proof of (i) and (ii)

One will not measure anything of E.

$\rightarrow$ One measures only observables which take the following form;

$$\hat{A}_{\rm S} \otimes \hat{1}_{\rm E}.$$

$\rightarrow$ This should be gauge-invariant, hence $\hat{A}_{\rm S}$ is gauge-invariant.

$\rightarrow$ $N_{\rm S}$ $(= N_1 + N_2)$ is conserved by the operation of $\hat{A}_{\rm S}$;

$$_1\langle N_1 \ell_1 |_2 \langle N_2 \ell_2 | \; \hat{A}_{\rm S} \; | N_2' \ell_2' \rangle_2 | N_1' \ell_1' \rangle_1 = \delta_{N_1+N_2, N_1'+N_2'} A^{N_1 \ell_1 N_2 \ell_2}_{N_1' \ell_1' N_2' \ell_2'}.$$

Hence, for any observable $\hat{A}_{\rm S}$ that will be measured,

$$\langle A_{\rm S} \rangle = {\rm Tr}_{\rm S} \left(\hat{\rho}_{\rm S} \hat{A}_{\rm S} \right)$$

$$= \sum_{N_1', \ell_1'} \sum_{N_2', \ell_2'} \sum_{N_1, \ell_1} \sum_{N_2, \ell_2} \delta_{N_1+N_2, N_1'+N_2'} C^{(1)}_{N_1' \ell_1'} C^{(2)}_{N_2' \ell_2'} C^{(2)*}_{N_2 \ell_2} C^{(1)*}_{N_1 \ell_1} \; A^{N_1 \ell_1 N_2 \ell_2}_{N_1' \ell_1' N_2' \ell_2'}$$

$$= {}_{\rm S}\langle \Phi_{\rm S} | \hat{A}_{\rm S} | \Phi_{\rm S} \rangle_{\rm S},$$

where

$$|\Phi_{\rm S}\rangle_{\rm S} = |\Phi^{(1)}\rangle_1 |\Phi^{(2)}\rangle_2, \quad |\Phi^{(k)}\rangle_k = \sum_{N_k, \ell} C^{(k)}_{N_k \ell} | N_k \ell \rangle_k. \qquad \blacksquare$$

AS and T. Miyadera, cond-mat/0102429.

There exist eigenstates $|\Phi\rangle_{\rm tot}$ of $\hat{N}_{\rm tot}$ with the following properties:

(i) The density operator $\hat\rho_{\rm S} \equiv {\rm Tr}_{\rm E}\left(|\Phi\rangle_{\rm tot\ tot}\langle\Phi|\right)$ satisfies $\hat\rho_{\rm S}^2 \neq \hat\rho_{\rm S}$.
But, $\hat\rho_{\rm S}$ is equivalent to a vector state $|\Phi_{\rm S}\rangle_{\rm S}$ ($\in \mathcal{H}_{\rm S}$) for any gauge-invariant observables in S.

(ii) $|\Phi_{\rm S}\rangle_{\rm S}$ is a product of vector states of $\rm S_1$ and $\rm S_2$;

$$|\Phi_{\rm S}\rangle_{\rm S} = |\Phi^{(1)}\rangle_1 |\Phi^{(2)}\rangle_2,$$

where $|\Phi^{(k)}\rangle_k$ is superposition of states with different charges,

$$|\Phi^{(k)}\rangle_k = \sum_{N_k,\,\ell} C^{(k)}_{N_k\ell} |N_k \ell\rangle_k.$$

$\rightarrow$ $\rm S_1$ and $\rm S_2$ are not entangled, but can have definite relative phase!

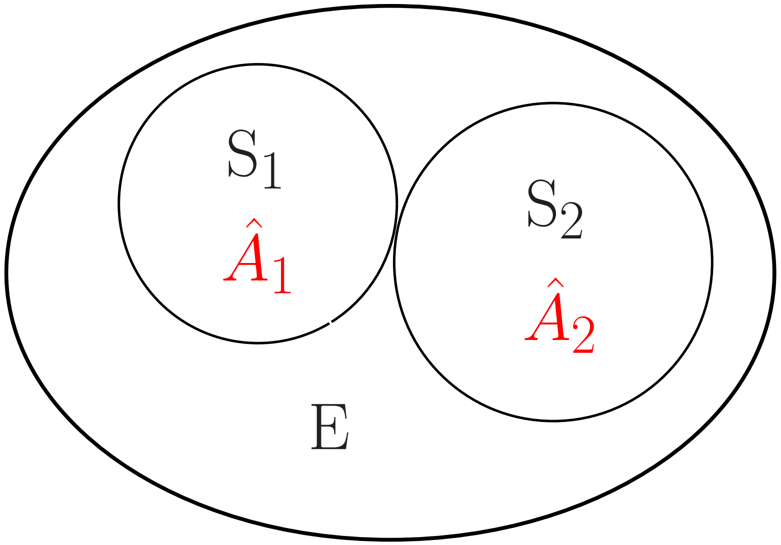
(iii) To each subsystem $\rm S_k$, one can associate $|\Phi^{(k)}\rangle_k$ and observables which are *not* necessarily gauge invariant in each subsystem.

(iv) In this association, $|\Phi^{(k)}\rangle_k$ is a pure state.

Proof of (iii) and (iv)

$\hat{A}_{\rm S}$ should be (a sum of) products of operators of each subsystems;

$$\hat{A}_{\rm S} = \hat{A}_1 \hat{A}_2 \text{ or } \hat{A}_1 \hat{A}_1' \text{ or } \hat{A}_2 \hat{A}_2'$$



Although $\hat{A}_{\rm S}$ is gauge invariant, each $\hat{A}_k$ is *not* necessarily gauge-invariant.

$\rightarrow$ For $|N_k \ell_k\rangle_k$ and $|N_k' \ell_k'\rangle_k$, there exists $\hat{A}_k$ s.t. ${}_k\langle N_k \ell_k| \, \hat{A}_k \, |N_k' \ell_k'\rangle_k \neq 0$

$\rightarrow$ $|\Phi^{(k)}\rangle_k$ is a pure state.

■

AS and T. Miyadera, cond-mat/0102429.

There exist eigenstates $|\Phi\rangle_{\rm tot}$ of $\hat{N}_{\rm tot}$ with the following properties:

(i) The density operator $\hat{\rho}_{\rm S} \equiv {\rm Tr}_{\rm E}\left(|\Phi\rangle_{\rm tot\ tot}\langle\Phi|\right)$ satisfies $\hat{\rho}_{\rm S}^2 \neq \hat{\rho}_{\rm S}$.
But, $\hat{\rho}_{\rm S}$ is equivalent to a vector state $|\Phi_{\rm S}\rangle_{\rm S}$ ($\in \mathcal{H}_{\rm S}$) for any gauge-invariant observables in S.

(ii) $|\Phi_{\rm S}\rangle_{\rm S}$ is a product of vector states of $\rm S_1$ and $\rm S_2$;

$$|\Phi_{\rm S}\rangle_{\rm S} = |\Phi^{(1)}\rangle_1 |\Phi^{(2)}\rangle_2,$$

where $|\Phi^{(k)}\rangle_k$ is superposition of states with different charges,

$$|\Phi^{(k)}\rangle_k = \sum_{N_k,\,\ell} C^{(k)}_{N_k \ell} |N_k \ell\rangle_k.$$

$\rightarrow$ $\rm S_1$ and $\rm S_2$ are not entangled, but can have definite relative phase!

(iii) To each subsystem $\rm S_k$, one can associate $|\Phi^{(k)}\rangle_k$ and observables which are *not* necessarily gauge invariant in each subsystem.

(iv) In this association, $|\Phi^{(k)}\rangle_k$ is a pure state.

So, we can play the game as ....

1. Decompose the system into S_1, S_2 and E:
   - S_2 can be an apparatus for measuring S_1.
   - You don't look at E.

     $\rightarrow$ Observables in S ($=S_1+S_2$) should be gauge-invariant.
2. But, observables in S_1 (or S_2) are *not* necessarily gauge invariant.
3. To S_1 and S_2, associate $|\Phi^{(1)}\rangle_1$ and $|\Phi^{(2)}\rangle_2$, respectively, which are superpositions of states with different charges.
4. Then,
   - $|\Phi^{(1)}\rangle_1$ and $|\Phi^{(2)}\rangle_2$ are pure states.
   - The state of S is the product state, $|\Phi_S\rangle_S \equiv |\Phi^{(1)}\rangle_1 |\Phi^{(2)}\rangle_2$.

For each subsystem, non-gauge invariant observables and a pure state which is a superposition of states with different charges.

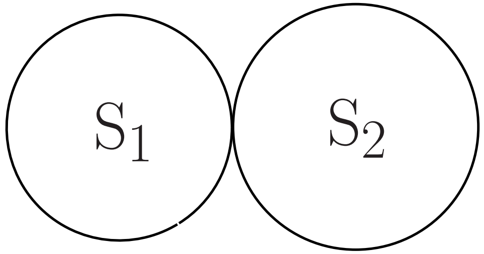
Note: Decomposition into S_1 and S_2 is insufficient. You need E! $\leftarrow$ realistic

Wrong points in the popular arguments

- 秩序変数 $\hat{\mathcal{O}}$ は、ゲージ不変でない：
  - BEC: $\hat{\mathcal{O}}(x) = \hat{\phi}(x)$
  - Superconductor : $\hat{\mathcal{O}}(x) = \hat{\psi}_\uparrow(x)\hat{\psi}_\downarrow(x)$
- ~~superselection rule により、N の定まった状態しか許されない~~.
- ~~ゆえに $\langle\hat{\mathcal{O}}(x)\rangle = 0$ であり、spontaneous symmetry breaking はない~~.
- しかし、Long-range order はある：

$$\langle\hat{\mathcal{O}}^\dagger(x)\hat{\mathcal{O}}(y)\rangle \not\to 0 \text{ as } |x-y| \to \infty \quad \text{spatially.}$$

- ~~Definite relative phase only when S_1 and S_2 are entangled~~.



$$J \propto \sin(\theta_1 - \theta_2)$$

- ?? ゆえに、~~$\langle\hat{\mathcal{O}}(x)\rangle \neq 0$ や~~ coherent state を仮定した議論は正しくない.

Then, what state is realized?

Analogy — transverse Ising model of finite size

$$\hat{H} = -J \sum_x \hat{\sigma}_Z(x) \hat{\sigma}_Z(x+1) - h \sum_x \hat{\sigma}_X(x)$$

Order parameter: $\hat{\mathcal{O}} = \sum_x \hat{\sigma}_Z(x)$ (total magnetization).

For $0 < h < {}^{\exists}h_c$, the exact ground state is

$$\begin{aligned} |G\rangle &= |\uparrow\uparrow\uparrow\cdots\rangle + |\downarrow\downarrow\downarrow\cdots\rangle + \text{small terms} \\ &= \text{cat state} + \text{small terms} \end{aligned}$$

- Unique
- Has the Z_2 symmetry

  $\rightarrow \langle \hat{\mathcal{O}} \rangle = 0$: symmtery is not broken.

But, macroscopically stable vacuum (or equilibrium) state $|vac\rangle$ should be

$$|\uparrow\uparrow\uparrow\cdots\rangle \quad \text{or} \quad |\downarrow\downarrow\downarrow\cdots\rangle \qquad \text{(ferromagnetic state)}$$

- Degenerate
- $\langle\hat{\mathcal{O}}\rangle = O(V)$: symmtery is broken.
- $E_{\rm vac} > E_{\rm G}$.

One of symmetry-breaking states is realized, although they have higher energies than the exact ground state (which is symmetric).

Why?

Because the latter does not have macroscopic stability.

General case: AS and T. Miyadera, PRL 89 (2002) 270403; BEC: PRL 85 (2000) 688.

A simplified example (with the Z_2 symmetry):

$$
\begin{aligned}
|G\rangle &= |\uparrow\uparrow\uparrow\cdots\rangle + |\downarrow\downarrow\downarrow\cdots\rangle \\
\text{measurement of } \hat{\sigma}_Z(1) &\Downarrow \quad \text{unstable} \\
|vac\rangle &= |\uparrow\uparrow\uparrow\cdots\rangle \\
\text{measurement of } \hat{\sigma}_Z(1) &\Downarrow \quad \text{stable} \\
|vac\rangle &= |\uparrow\uparrow\uparrow\cdots\rangle \\
\text{measurement of } \hat{\sigma}_X(1) &\Downarrow \quad \text{stable, macroscopically} \\
|\text{single-spin excitation on } vac\rangle &= |\rightarrow\uparrow\uparrow\cdots\rangle
\end{aligned}
$$

Theorem (a simplified version): The symmteric ground state with a long-range order is unstable against local measurement of $\hat{\mathcal{O}}(x)$, i.e., does not have macroscopic stability.

Energy is not sufficient to determine the vacuum; stability is important!

Where do you encounter $E_{\text{vac}} > E_{\text{G}}$?

This often occurs, in the absence of a symmetry-breaking field, when

$$[\hat{H},\ \hat{\mathcal{O}}] \neq 0 \quad (\hat{\mathcal{O}} = \text{order parameter}).$$

- **Antiferro magnet**

$$\hat{\mathcal{O}} = \sum_x (-1)^x \hat{\sigma}_Z(x) \quad \text{(staggered magnetization)}.$$

A symmetry-breaking field $\vec{h}(x) = (-1)^x \vec{h}_Z$ is highly artificial.

- **$U(1)$ gauge symmetry breaking**
  - BEC: $\hat{\mathcal{O}} = \int \hat{\phi}(x) dx$
  - Superconductor : $\hat{\mathcal{O}} = \int \hat{\psi}_\uparrow(x) \hat{\psi}_\downarrow(x) dx$

c.f. For **simple ferromagnets**, $E_{\text{G}} = E_{\text{vac}}$ because

$$[\hat{H},\ \hat{\mathcal{O}}] = 0 \text{ for } \hat{\mathcal{O}} = \sum_x \hat{\sigma}_Z(x).$$

A symmetry-breaking field $\vec{h}(x) = \vec{h}_Z$ is natural.

$E_{vac} - E_G$ for $U(1)$ gauge symmetry breaking systems

(for equal $\langle N \rangle$, a unifrom system of large V, with PBC)

$$\text{ground state} : \hat{H}|G\rangle = E_G|G\rangle, \quad \hat{N}|G\rangle = N|G\rangle.$$
$$\text{vacuum} : E_{vac} = \langle \hat{H} \rangle, \quad \langle \hat{N} \rangle = N, \quad \delta N^2 \equiv \langle (\Delta \hat{N})^2 \rangle \neq 0.$$

Thermodynamics requires

$$E_{vac} - E_G = o(V) \text{ or, more strongly, } E_{vac} - E_G = O(1)?$$

cf. For breaking of Z_2 symmetry (Horsch and von der Linden, 1988);

$$E_{vac} - E_G \leq O(1/V).$$

For short-range interactions (AS and T. Miyadera, PRE 64 (2001) 056121);

$$E_{vac} - E_G \geq \mu' \frac{\delta N^2}{2V} + \frac{o(V)}{V}, \quad \mu' \equiv \frac{\partial \mu}{\partial n} = O(1) > 0.$$

When $\delta N^2 = O(\langle \hat{N} \rangle) = O(V)$,

$$E_{vac} - E_G \geq \text{a positive constant of } O(1).$$

How was such a strict inequality derived?

Fully quantum mechanical derivation is hard; the best result is

$$E_{vac} - E_G \leq O(\sqrt{V}) \to \infty \text{ when } \delta N^2 = O(V),$$

for a specific model. (T. Koma and H. Tasaki, J. Stat. Phys. 76 (1994) 745)

Our inequality is more strict and universal;

$$E_{vac} - E_G \geq \mu' \frac{\delta N^2}{2V} + \frac{o(V)}{V}$$
$$= \text{a positive constant of } O(1) \text{ when } \delta N^2 = O(V).$$

We have utilized quantum mechanics and thermodynamics:

$$\text{quantum mechanics}: \hat{H}|N,\ell\rangle = E_{N,\ell}|N,\ell\rangle, \quad \hat{N}|N,\ell\rangle = N|N,\ell\rangle,$$
$$|G\rangle = |N,G\rangle, \quad |vac\rangle = \sum_{N,\ell} C_{N,\ell}|N,\ell\rangle.$$
$$\text{thermodyn. extensivity}: E_{N,\mathrm{G}} = V\left[\epsilon(N/V) + o(V)\right] \quad (S \to 0),$$
$$\text{thermodyn. stability}: \mu'(n) \equiv \epsilon''(n) = V\frac{\partial^2}{\partial N^2}E_{N,\mathrm{G}} = O(1) > 0,$$

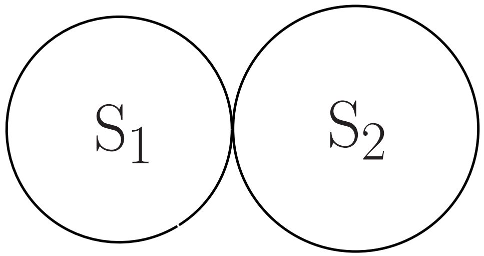
Such powers of thermodynammics are stressed in 清水「熱力学の基礎」(東大出版会)

Instability of $|G\rangle = |N, G\rangle$

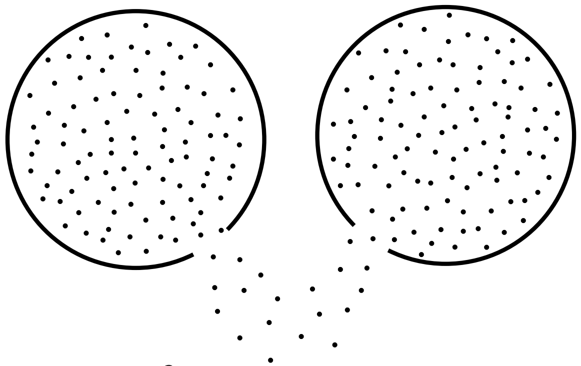
This is the symmteric ground state with a long-range order.

$\Downarrow$ above-mentioned theorem

Does not have macroscopic stability.
Unstable against local measurement of $\hat{\mathcal{O}}(x) = \hat{\phi}(x)$.



$$J \propto \sin(\theta_1 - \theta_2)$$



interference pattern

What state is realized as a vaccum?

Theorem : AS and T. Miyadera, PRL 89 (2002) 270403; Y. Matsuzaki and AS, 2006
A state with the cluster property is stable against any local measurement, i.e., has macroscopic stability.

So, the **conditions for the vacuum state** are summarized as;

1. Energy is low enough:

$$E_{vac} - E_G = o(V) \text{ or, more strongly, } E_{vac} - E_G = O(1)?$$

2. Macroscopic stability (i.e., cluster property).
3. Compatibe with other physical situations of each system.

c.f. Nucleus
Large energy barrier against removing a particle
$\longrightarrow$ ground state with fixed N; BCS state is a useful convention

A candidates for a vacuum state for short-range interactions

'Coherent state of interacting bosons'

AS and J. Inoue, PRA 60 (1999) 3204; AS and J. Inoue, JPSJ 71 (2002) 56

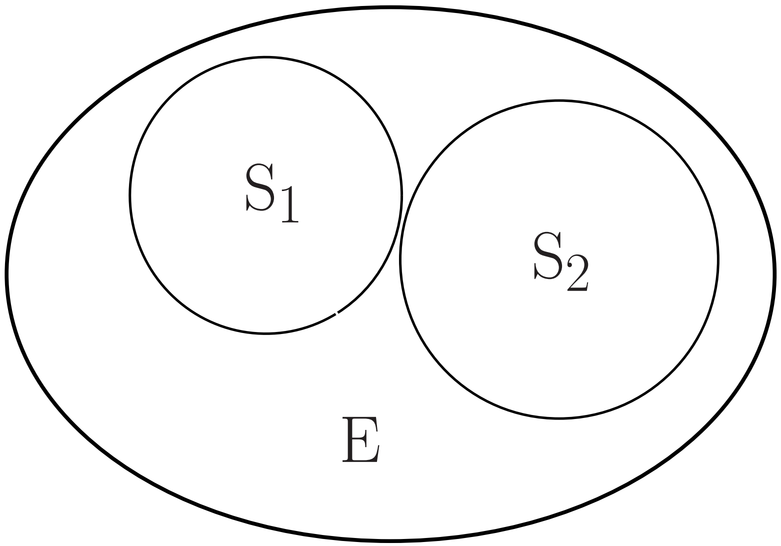
$$|\alpha, G\rangle = e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} |N, G\rangle \quad (|\alpha|^2 = \langle N \rangle)$$

- Symmery is broken: $\langle \hat{\phi}(x) \rangle = \sqrt{Z} \frac{\alpha}{\sqrt{V}} = O\left(\sqrt{N/V}\right) = O(1)$.

  $(Z = O(1), 0 < Z < 1$ for interacting bosons$)$
- Cluster property
- $\delta N^2 = \langle \hat{N} \rangle = O(V) \ \rightarrow \ E_{\alpha,G} - E_G = O(1)$.
- Stable against leakage of particles

Therefore .....

When particles can flow between subsystems, $\leftarrow$ realistic
the coherent state of interacting bosons $\leftarrow |\alpha, G\rangle$
would be realized in each of S_1 and S_2



- [size of E] $\gg$ [size of S].
- One will not measure observables in E.
- S_2 can be (a part of) an apparatus for measuring S_1.

$$|\Phi_S\rangle_S = |\alpha_1, G\rangle_1 |\alpha_2, G\rangle_2$$

- $\dfrac{\alpha_1}{\sqrt{V_1}} = \dfrac{\alpha_2}{\sqrt{V_2}}$ at equilibirum. If not, finite current.
- Similar results when $S = S_1 + S_2 + S_3 + \cdots$.

$\langle \hat{\mathcal{O}}(x) \rangle \neq 0$やcoherent stateを仮定した議論は、coherent state of interacting bosonsに置き換えれば正しくなる！

Superconductors — long-range interactions

If we regard a Cooper pair as a boson, a trivial extension of the short-range case gives (AS, talk presented in 2003)

$$E_{vac} - E_G \geq \mu' \frac{\delta N^2}{2V} + K \frac{\delta N^2}{V^{1/3}} + \frac{o(V)}{V}.$$

Thermodynamics requires

$$E_{vac} - E_G = o(V) \text{ or, more strongly, } E_{vac} - E_G = O(1)?$$

Therefore, for large V,

- $|\alpha, G\rangle$ would not be realized in superconductors, because $\delta N^2 = O(V)$.
- States with $\delta N^2 \leq O(V^{1/3})$ would be realized.
- But, $|N, G\rangle$ is macroscopically unstable (for large V).

Then, what state is realized?

A candidates for a vacuum state for long-range interactions

'Number-phase squeezed state of interacting bosons'

AS and J. Inoue, PRA 60 (1999) 3204; AS and J. Inoue, JPSJ 71 (2002) 56

$$|N, \zeta, G\rangle = \text{constant} \times \sum_{M=0}^{N} \frac{\zeta^{*(N-M)}}{\sqrt{(N-M)!M!}} |M, G\rangle \quad (N - |\zeta|^2 = \langle N \rangle)$$

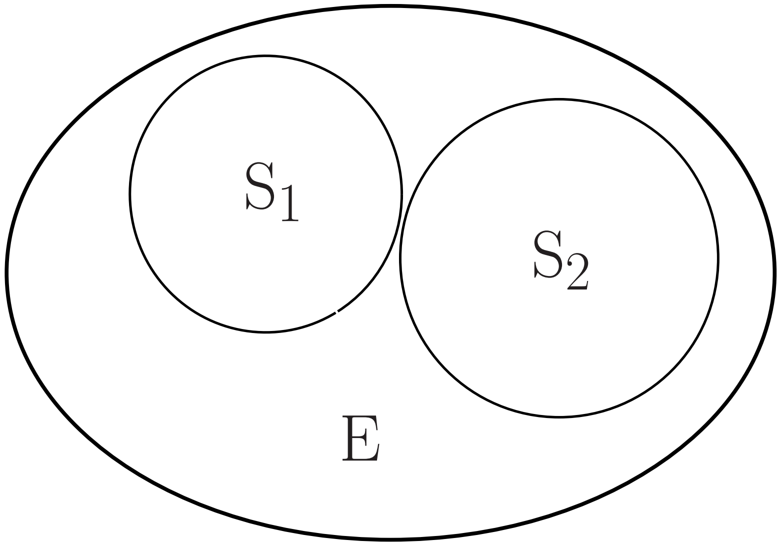
If we take $|\zeta|^2 = O(V^{1/3})$ ($\to \infty$ as $V \to \infty$), then

- Symmery is broken: $\left| \langle \hat{\phi}(x) \rangle \right| \simeq \sqrt{Z} \sqrt{\frac{N}{V}} = O\left(\sqrt{N/V}\right) = O(1).$

  ($Z = O(1), 0 < Z < 1$ for interacting bosons)
- (Probably) Cluster property.
- $\delta N^2 = |\zeta|^2.$
- For superconductors, $E_{N,\zeta,G} - E_G = O(1).$

Therefore ...

When particles can flow between subsystems, $\leftarrow$ realistic
the number-phase squeezed state of interacting bosons would be realized in each of $\mathbf{S}_1$ and $\mathbf{S}_2$



- [size of E] $\gg$ [size of S].
- One will not measure observables in E.
- S_2 can be (a part of) an apparatus for measuring S_1.

$$|\Phi_S\rangle_S = |N_1, \zeta_1, G\rangle_1 |N_2, \zeta_2, G\rangle_2$$

Similar results when $S = S_1 + S_2 + S_3 + \cdots$.

$\langle \hat{\mathcal{O}}(x) \rangle \neq 0$ や coherent state を仮定した議論は、number-phase squeezed state of interacting bosons に置き換えれば正しくなる！

Different states are realized under different conditions

- Small superconductor

  Large energy barrier against changing N

  $\rightarrow$ ground state with a fixed N is stable and realized

- etc.

So far, so good .... but ....!!!

In inifinite systems, a vacuum is assumed to be time-independent.

In finite systems,

- $|N, G\rangle$: eigenstate of $\hat{H}$ $\longrightarrow$ no time evalution if perturbation is absent.
  But, discarded because macroscopically unstable.
- $|\alpha, G\rangle$: would be realized because macroscopically stable.
  But, not eigenstate of $\hat{H}$ $\longrightarrow$ time evolution even if perturbation is absent!

How can $|\alpha, G\rangle$ be consistent with

- a vacuum of infinite systems?
- thermodynamics, where the equilibrium state is time-independent?

AS and T. Miyadera, PRE 64 (2001) 056121

Although $t_{\rm clps}$ would be finite for finite V, it is sufficient that

$$t_{\rm clps} \to \infty \text{ as } V \to \infty.$$

However, a naive calculation gives;

$$|\alpha, G\rangle = e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} |N, G\rangle \;\Rightarrow\; \delta N = \langle N \rangle$$

$$E_{N+\delta N, G} - E_{N,G} = \mu \delta N + \mu' \frac{(\delta N)^2}{2V} + \cdots$$

$$\Rightarrow\; t_{\rm clps}^{\rm wf} \sim 1/\mu \delta N = 1/O(N) \to 0???$$

However, the factor $\mu \delta N$ can be absorbed into

$$\alpha \to \alpha e^{-i\mu t} \;\Rightarrow\; \text{Josephson effect}$$

If interaction were absent, this solves the problem (well known).

However, $\mu' > 0$ because of interactions, so

$$t_{\rm clps}^{\rm wf\,\prime} \sim V/\mu'(\delta N)^2 = O(V/N) = O(1).$$

The wavefunction collapses in such a short time!!

However, this does not necessarily mean that expectation values of observables of interest alter in this time scale.

For an observable that is proportional to a field operator,

$$t_{\rm clps}^{\rm obs} \sim V/(\mu' \delta N) = O(\sqrt{V}) \to \infty.$$

For an observable that is a polynomial of degree M of field operators,

$$t_{\rm clps}^{\rm obs} = O(\sqrt{V}/M).$$

Therefore, if M is independent of V,

$$t_{\rm clps}^{\rm obs} = O(\sqrt{V}) \to \infty.$$

Consistent with

- a vacuum of infinite systems.
- thermodynamics, where the equilibrium state is time-independent.

Summary and Conclusions

- By considering the environment, we can associate a pure state and non-gauge invariant observables to each subsystem.
- A vacuum state of a finite system is not necessarily the ground state.
- The conditions for the vacuum state are
  1. Energy is low enough:
  $$E_{vac} - E_G = o(V) \text{ or, more strongly, } E_{vac} - E_G = O(1)?$$
  2. Macroscopic stability (i.e., cluster property).
  3. Compatibe with other physical situations of each system.
- Candidates for the realized vacua, for short-range interactions and for long-range interactions.
- When $|vac\rangle \neq |G\rangle$, the state vector $|vac\rangle$ evolves quickly.
- However, if we look only at observables that are low-order polynomials of field operators, their expactation values evolve slowly enough.